Optical analysis of spatially periodic patterns in nematic liquid crystals: Diffraction and shadowgraphy

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Optical methods are most convenient for analyzing spatially periodic patterns with wave vector $\boldsymbol{q}$ in a thin layer of a nematic liquid crystal. In the standard experimental setup a beam of parallel light with a "short" wavelength $\lambda \ll 2\pi/q$ passes the nematic layer. Recording the transmitted light the patterns are either directly visualized by shadowgraphy or characterized more indirectly by the diffraction fringes due to the optical-grating effects of the pattern. In this work we present a systematic short-wavelength analysis of these methods for the commonly used planar orientation of the optical axis of liquid crystal at the confining surfaces. Our approach covers general three-dimensional experimental geometries with respect to the relative orientation of $\boldsymbol{q}$ and of the wave vector $\boldsymbol{k}$ of the incident light. In particular, we emphasize the importance of phase-grating effects, which are not accessible in a pure geometric optics approach. Finally, as a by-product we present also an optical analysis of convection rolls in Rayleigh-Bénard convection, where the refraction index of the fluid is isotropic in contrast to its uniaxial symmetry in nematic liquid crystals. Our analysis is in excellent agreement with an earlier physical optics approach by Trainoff and Cannell [Phys. Fluids **14**, 1340 (2002)], which is restricted to a two-dimensional geometry and technically much more demanding.

PACS number(s): 42.70.Df, 78.20.Bh, 61.30.−v

I. INTRODUCTION

Increasing an external stress on a homogeneous fluid layer leads typically to a spontaneous generation of spatially periodic structures (patterns) in the plane of the layer [1]. The structures are characterized by a wave vector $\boldsymbol{q}$ and a certain amplitude, which depends on the amount of stress. Corresponding periodic structures are induced into the refraction index of the fluid layer, which thus acts as an optical grating when illuminated, for instance, with a parallel light beam with wavelength $\lambda \ll 2\pi/q$. The transmitted light may be analyzed in terms of the arising diffraction fringes. They give insight into the intensity of the Fourier modes representing the periodicity of the planforms and more indirectly into the mechanism driving the pattern forming instabilities. Alternatively, shadowgraphy is often applied to visualize directly the periodically distorted fluid layer. An important paradigm is Rayleigh-Bénard convection (RBC) driven by a temperature gradient [2], where the common array of convection rolls is mapped to a sequence of black-and-white stripes in shadowgraphy [3].

In this paper, however, we refer mainly to patterns in nematic liquid crystals, which are anisotropic uniaxial fluids. The preferred direction (roughly speaking the mean orientation of the non-spherical molecules in the nematic phase) is described by the director $\hat{\mathbf{n}}$ with $\hat{\mathbf{n}}^2 = 1$, which also determines the local optical axis. The nematic layer of thickness d and large lateral extension is confined between two coplanar glass plates (parallel to the xy plane). They are specially treated to enforce a fixed director orientation $\hat{\mathbf{n}} = \boldsymbol{n}_0$ at the two surfaces at $z = 0,d$. We consider exclusively the so-called *planar* director configuration where $\boldsymbol{n}_0$ points along the x axis (parallel to the unit vector $\hat{\boldsymbol{x}}$). In the basic state a uniform orientation of $\hat{\mathbf{n}} = \boldsymbol{n}_0$ is then induced throughout the whole

nematic layer, corresponding to a minimum of the orientational elastic free energy. A thoroughly studied pattern forming instability of the basic state is the electroconvection instability (EC), where the glass plates are coated in addition with thin transparent electrodes to apply an ac voltage $U = U_0 \cos(\Omega t)$ across the layer. For a voltage amplitude U_0 above a certain threshold U_c a stripe pattern appears in EC [4]. However, the application of magnetic fields, temperature gradients, or shearing the nematic layer also leads easily to patterns (see, for instance, [5]). Revealing the character and the mechanisms of the instabilities requires relating the director distortions and the corresponding optical signals.

The theory of shadowgraphy has been developed in several steps over the years. At first geometric (ray) optics has been applied to RBC in Ref. [6]. There a simple model was proposed to describe the deflection of the incoming light rays towards the optically denser (cold) regions of the RBC convection patterns. The model makes use of the fact that the scale of the spatial variations of the refraction index in the fluid layer (of the order of 0.5 cm in typical RBC experiments) is much larger than the light wavelength $\lambda \approx 0.6 \ \mu$m used. Since the effects of diffraction are neglected, this theory predicts divergent intensities (caustics) in pictures recorded at a certain level $z = z_F$ above the cell. In Ref. [6] z_F is treated as an adjustable parameter and the theoretical pictures calculated for some distance $z < z_F$ above the cell, look very similar to the experimental ones. The analysis in RBC on the basis of ray optics has been considerably refined by Rehberg and co-workers Refs. [7,8] by calculating the ray paths using Fermat's principle. In addition, they considered also shadowgraphy for optically anisotropic nematics on the example of EC pattern (see also [9]). Here the scale of the spatial variations of the refraction index is typically governed by the thickness of the nematic layer (typically $10 \ \mu$m $\lesssim d \lesssim 100 \ \mu$m) and thus is still considerably larger than $\lambda \approx 0.6 \ \mu$m of the light used. Roughly speaking, in Refs. [7,8] a theoretical description of z_F as a function of the pattern amplitudes and the two nematic

refraction indices is achieved. Focusing in experiments to positions $z < z_F$ above the cell, the contrast between minimal and maximal amplitudes in shadowgraph pictures seems to agree quite well with the theoretical predictions.

The most comprehensive theoretical treatment of shadowgraphy so far has been presented by Trainoff and Cannell [10], where some problematic aspects of the prior analyses become evident [11]. In geometric optics only the amplitude-grating effects of the patterns but not their phase-grating contributions are taken into account. They determine, for instance, the weight of higher harmonics with wave vector $2\boldsymbol{q}$ in shadowgraph pictures and are essential for the interpretation of patterns driven by thermal fluctuations below onset of convection (see, e.g., Refs. [10,12]). Furthermore, geometric optics is not used outside the fluid layer as in Refs. [7,8] and the propagation and interference of the light waves when leaving the fluid layer are treated rigorously. As a result, the electromagnetic patterns with maximal contrast above the fluid layer reoccur at regular distances along the z axis perpendicular to the layer plane. The corresponding period is given as $8\pi^2/(q^2\lambda)$ and does not depend on the pattern amplitudes. This is in distinct contrast to the caustics in geometric optics. Here the position of shadowgraph pictures with maximal contrast above the layer is determined by z_F, which even diverges when the pattern amplitude approaches zero. It is noteworthy that the periodicity of the electromagnetic field along the z direction was already described almost 200 years ago by Talbot [13], who studied light incident on a periodic diffraction grating. The intriguing transition from the periodic sequence of the finite-intensity patterns along the z axis to caustics in the limit $\lambda \to 0$ has been discussed recently in Ref. [14].

In [10] shadowgraphy in RBC has been discussed in detail, while EC is only briefly touched. A more detailed recent investigation [15] on EC patterns is, on one hand, devoted to the theoretical description of the diffraction spots (on the basis of phase grating). The main theoretical predictions are consistent with corresponding experiments. This paper contains also references to earlier investigations of the problem. In particular, in Ref. [16] the importance of phase grating had been already stressed.

The *physical optics* approach in Refs. [10,15] is, however, quite complicated to use since amplitude- and phase-grating effects are still treated separately. Furthermore the analysis is restricted to a special, though often utilized, two-dimensional (2D) geometry, where $\hat{\mathbf{n}}$ lies in plane spanned by the wave vector $\boldsymbol{q}$ and the polarization of the plane wave transversing the nematic layer. The goal of the present work is a systematic, more transparent analysis of the problem on the basis of the common short-wavelength approximation in theoretical optics [17]. The starting point is a systematic expansion of the solutions of the Maxwell equations in terms of the small parameter $\lambda q/(2\pi)$. We have benefited considerably from an analysis of light propagation in nematic fluid layer presented in Ref. [18]. As a result, for the first time diffraction and shadowgraphy for patterns in nematics are thus described quasianalytically for quite general configurations of the incident light wave (polarization, wave vector) and of the orientation of the director. In the case of dissipative convection rolls as in EC the presence of fluid flow plays an essential role. The wave vector $\boldsymbol{q}$, characterizing the pattern periodicity, is

then often parallel to the basic planar director configuration $\boldsymbol{n}_0 \equiv \hat{\boldsymbol{x}}$ or includes only a small angle with $\boldsymbol{n}_0$. Alternatively, one finds so called equilibrium transitions when the minima of orientational free energy exchange in the presence of electric or magnetic fields, where $\boldsymbol{q} \perp \boldsymbol{n}_0$. Interesting examples are the flexoelectric stripe patterns [19] driven electrically or the splay-twist Freedericksz pattern [20] driven either by electric or magnetic fields.

The paper is organized as follows. In Sec. II we first review briefly the salient features of the pattern forming instabilities in nematics. Furthermore, we sketch the theoretical background of light propagation in nematics, partly in order to fix our notation. Section III deals with the short-wavelength approximation of the Maxwell equations, which is appropriate to describe optics in media with a slowly varying refraction index in space. Based on this approximation we rederive and confirm in Sec. IV the results of [10] in their 2D geometry. The following two sections contain the main results of the paper, on which one might concentrate on a first reading. In Sec. V we discuss the optics of 3D patterns. The treatment covers, in particular, the case where $\boldsymbol{q} \perp \boldsymbol{n}_0$, which cannot be described by pure geometric optics. Section VI is devoted to a general discussion and summary of our results. The paper concludes with some final remarks also on future perspectives in Sec. VII. In several appendixes, we provide details of our calculations. In particular, Appendix F is devoted to our complementary optical analysis of roll patterns in RBC.

II. GENERAL THEORETICAL BACKGROUND

In this section some well known basic facts on pattern forming instabilities in nematics are briefly summarized. Furthermore, since the patterns are analyzed by optical methods, we mention briefly the standard description of light propagation in uniaxial materials mainly in order to fix the notation.

A. Pattern forming instabilities in nematics

At first we address briefly some main features of the typical stripe patterns in nematic layers in the planar configuration defined in the Introduction. For a sufficiently strong external stress the basic state is destabilized. The director develops a distortion $\Delta\boldsymbol{n}(x,y,z)$ of $\boldsymbol{n}_0$ for $0 < z < d$, which is periodic in the xy plane. The periodicity is characterized by a critical wave vector $\boldsymbol{q}$; i.e., $\Delta\boldsymbol{n}$ can be represented as a Fourier series in terms of $e^{i\boldsymbol{q}\cdot\boldsymbol{x}}$ with $\boldsymbol{x} = (x,y)$. At the confining plates, however, the director orientation remains fixed (strong anchoring): $\Delta\boldsymbol{n}(\boldsymbol{x},z=0,d) = 0$. In this paper we concentrate on the weakly nonlinear regime just above the onset of the pattern forming instability, where the director distortion $\Delta\boldsymbol{n}(\boldsymbol{x},z)$ is small. It is convenient to introduce the following decomposition of $\Delta\boldsymbol{n}$:

$$\Delta\boldsymbol{n}(\boldsymbol{x},z) = -\delta n_x(\boldsymbol{x},z)\,\hat{\boldsymbol{x}} + \theta_m\delta\boldsymbol{n}(\boldsymbol{x},z), \quad \text{with}$$
$$\delta\boldsymbol{n} = (0,\delta n_y,\delta n_z) \text{ and } \delta\boldsymbol{n}\cdot\hat{\boldsymbol{x}} = 0. \tag{1}$$

Exploiting the normalization condition $\hat{\mathbf{n}}^2 = 1$ with $\hat{\mathbf{n}} = \boldsymbol{n}_0 + \Delta\boldsymbol{n}(x,y,z)$ determines δn_x in terms of $\delta\boldsymbol{n}$:

$$\hat{\mathbf{n}}^2 = [(1 - \delta n_x)\hat{\boldsymbol{x}} + \theta_m\delta\boldsymbol{n}]^2 = 1; \quad \text{thus,}$$

$$\delta n_x = \tfrac{1}{2}\theta_m^2(\delta \boldsymbol{n})^2 + O\!\left(\theta_m^4\right). \tag{2}$$

The amplitude θ_m measures the maximal local tilt angle of $\hat{\mathbf{n}}$ with respect to the xy plane, i.e., the out-of-plane distortion of $\boldsymbol{n}_0$. It has turned out in many cases that the director field is surprisingly well described already within a "one-mode" approximation of the form

$$\hat{\mathbf{n}}(\boldsymbol{x},z) = \left(1 - \tfrac{1}{2}\theta_m^2(\delta \boldsymbol{n})^2,0,0\right) + \theta_m\delta\boldsymbol{n},$$
$$\delta\boldsymbol{n} = \vartheta(z)(0,a_y\sin(\boldsymbol{q}\cdot\boldsymbol{x}),\cos(\boldsymbol{q}\cdot\boldsymbol{x})), \tag{3}$$

with $\vartheta(z) = \sin(\pi z/d)$ to fulfill the boundary condition $\hat{\mathbf{n}}(\boldsymbol{x},z=0,d) = \boldsymbol{n}_0$. The amplitude a_y of the twist distortion (δn_y) of the director is small for dissipative patterns like in EC and of the order one for the equilibrium patterns like the flexoelectric ones. The standard optical analysis reflects thus directly the wave vector $\boldsymbol{q}$, the amplitudes θ_m, a_y, and two refraction indices n_e and n_o of uniaxial materials (see also following Sec. II B).

For convenience we choose the well studied EC as a representative example to show some details, but our analysis makes no particular use of this special case. The experimental setup consists of an extended nematic fluid layer of small thickness $d \lesssim 100\ \mu$m oriented parallel to the xy plane, where the lateral extension are much larger than d. The amplitude coefficients θ_m and a_y are in general periodic functions in time governed by the ac frequency (up to 10^4 Hz in some experiments) of the applied voltage. In many cases the coefficients are already well described by their temporal average and thus considered to be constant. In any case the time dependence of the amplitude can be treated in an adiabatic approximation for optical methods, since the frequency ω of the monochromatic light waves to probe the patterns is by many orders of magnitude larger than the ac frequency Ω. Convection sets in at a critical amplitude $U_0 = U_c$ (typically some volts), where $\boldsymbol{q}$ is determined by the critical wave vector $\boldsymbol{q}_c$ with $|\boldsymbol{q}_c|$ of the order of π/d. These critical data as well as the detailed director configuration depend on the material parameters of the specific nematic material, the cell thickness, and the ac frequency Ω. They are available from a linear stability analysis of the spatially homogeneous basic state, which has been extensively carried through in the last two decades (see, e.g., Refs. [21,22]). The EC instability, as most pattern forming instabilities in nematics, is supercritical, which means that for $U_0 \gtrsim U_c$ the amplitude θ_m grows continuously with the reduced control parameter $\epsilon \equiv (U_0^2 - U_c^2)/U_c^2$ like $\sqrt{\epsilon}$. For the nematic material MBBA, which has been used in the majority of EC experiments, the proportionality factor is about 80° [21], such that $\epsilon = 0.1$ corresponds to θ_m about 25°.

To analyze the pattern forming instabilities driven by other external stresses alluded to above, one has to follow the same strategy as in EC. At first the critical stress value like U_c in EC and the critical wave vector $\boldsymbol{q}_c$ together with the distorted director configuration are determined in the framework of a linear stability analysis. Then a weakly nonlinear analysis has to be used to calculate the amplitude of the director distortion as a function of the stress parameter. It should be realized that the range of validity of a weakly nonlinear analysis is, in general, confined to small amplitudes θ_m. Increasing the external stress and thus θ_m leads to secondary instabilities (see,

e.g., [23,24]). The resulting patterns are typically characterized by the appearance of spatial periodicities with wave vectors not parallel to $\boldsymbol{q}_c$. That the growth of the out-of-plane distortion of $\propto\theta_m$ with increasing ϵ must be limited is obvious, since, for instance, the stabilizing orientational elastic energy grows. For the nematic material MBBA, which has been used in the majority of EC experiments, the secondary instabilities set in for angles θ_m between 20° and 30°. The often complex spatiotemporal pattern developing due to such secondary instabilities are outside the scope of this paper.

B. Light propagation in nematics

In the following we review briefly the propagation of monochromatic light waves with circular frequency ω in a nematic liquid crystal, which has uniaxial optical symmetry. In general, we follow closely the notations of Born and Wolf [17]. The starting point comprises the general Maxwell equations (in cgs units), where the factor $e^{-i\omega t}$ has been split off,

$$\nabla \times \boldsymbol{H} = -ik_0\,\boldsymbol{\varepsilon}\cdot\boldsymbol{E}, \quad \nabla \times \boldsymbol{E} = ik_0\,\boldsymbol{B}, \tag{4a}$$
$$\nabla \cdot \boldsymbol{D} = 0, \quad \nabla \cdot \boldsymbol{B} = 0, \tag{4b}$$

with $k_0 = \omega/c$ and c the vacuum speed of light. All fields depend, in general, on $\boldsymbol{r} = (\boldsymbol{x},z)$. The constitutive equations which connect the dielectric displacement $\boldsymbol{D}$ to the electric field $\boldsymbol{E}$ and the magnetic induction $\boldsymbol{B}$ to the magnetic field $\boldsymbol{H}$ are given as

$$\boldsymbol{D} = \boldsymbol{\varepsilon}\cdot\boldsymbol{E}, \quad \boldsymbol{B} = \mu\boldsymbol{H}. \tag{5}$$

The magnetic permeability μ is a scalar which can be safely put to one for our materials. The optical dielectric tensor $\boldsymbol{\varepsilon}$ of the uniaxial nematics is given as

$$\boldsymbol{\varepsilon} = \epsilon_\perp\mathbf{I} + (\epsilon_\parallel - \epsilon_\perp)\hat{\mathbf{n}}\otimes\hat{\mathbf{n}}. \tag{6}$$

Here we have introduced the standard definition of the tensor (dyadic) product $\boldsymbol{a}\otimes\boldsymbol{b}$ of two vectors $\boldsymbol{a},\boldsymbol{b}$ with the components a_ib_j, $i,j = x,y,z$; $\mathbf{I}$ denotes the unit matrix. The frequency dependent dielectric constants $\epsilon_\perp,\epsilon_\parallel$ are taken at the frequency ω. They are considered to be real; i.e., light absorption plays no role. The case of optically isotropic materials like glass or air are characterized by $\epsilon_\perp = \epsilon_\parallel \equiv \bar{\epsilon}$.

After eliminating $\boldsymbol{B}$ from Eq. (4a) we arrive at

$$\text{rot rot}\,\boldsymbol{E} = \Delta\boldsymbol{E} - \nabla(\nabla\cdot\boldsymbol{E}) = k_0^2\mu\,\boldsymbol{\varepsilon}\cdot\boldsymbol{E}. \tag{7}$$

The corresponding elimination of the electric field $\boldsymbol{E}$ is possible as well, which leads to

$$\text{rot}[\boldsymbol{\varepsilon}^{-1}\text{rot}\boldsymbol{B}] = k_0^2\mu\boldsymbol{B}. \tag{8}$$

Plane-wave solutions. In the case of constant $\hat{\mathbf{n}}$ the Maxwell equations (4) allow for plane wave solutions (characterized by a wave vector $\boldsymbol{k} = k_0\boldsymbol{k}'$), which play an important role in the following. For instance, the electric field $\boldsymbol{E}$ has the representation

$$\boldsymbol{E} = \boldsymbol{E}_0\exp[ik_0\boldsymbol{k}'\cdot\boldsymbol{r}], \quad \text{with } \boldsymbol{r} = (\boldsymbol{x},z) \text{ and } k_0 = \frac{\omega}{c}. \tag{9}$$

The amplitude $\boldsymbol{E}_0$ is a constant vector. Note that the use of nondimensionalized wave vectors (indicated by a dash) like $\boldsymbol{k}' = \boldsymbol{k}/k_0$ in Eq. (9) turns out to be very convenient in the

following. For instance, the *refraction index* n_{rf} fulfills relation $(k')^2 = (n_{rf})^2$.

Representing the other fields in analogy to Eq. (9) with amplitudes D_0, B_0, H_0 we arrive from Eqs. (9) at

$$k' \times H_0 = -D_0 = \boldsymbol{\varepsilon} \cdot E_0, \quad k' \times E_0 = \mu H_0, \quad (10a)$$

$$k' \cdot D_0 = 0, \quad k' \cdot B_0 = 0. \quad (10b)$$

According to Eqs. (10) the vectors k', D_0, H_0 are pairwise orthogonal. Furthermore, E_0 is orthogonal to H_0, i.e., lies in the plane of k', D_0. It is easy to see that Eq. (7) reduces for the plane wave E used in Eq. (9) to

$$\{(k')^2 - k' \otimes k' - \mu[\epsilon_\perp + (\epsilon_\parallel - \epsilon_\perp)\hat{\mathbf{n}} \otimes \hat{\mathbf{n}}]\} \cdot E_0$$
$$\equiv M[k'] \cdot E_0 = 0. \quad (11)$$

For definiteness we concentrate in the following on the typical (planar) experimental setup of a nematic layer parallel to the xy plane with a constant $\hat{\mathbf{n}} = \hat{x}$ sandwiched by the coplanar glass plates. Let a monochromatic plane wave enter this configuration from air with the wave vector $k' = k'_\parallel + k'_z \hat{e}_z, k'_z > 0$. At each interface the wave is reflected (k'_z reverses sign) and diffracted; i.e., it propagates above the interface with a modified k'_z. According to Snellius's law the in-plane components of the wave vector k' in the various material layers are equal. The z components of k' are determined by the respective refraction indices, n_{rf}, i.e., by the condition $(k')^2 = (k'_\parallel)^2 + (k'_z)^2 = n_{rf}^2$.

In isotropic media where $\epsilon_\parallel = \epsilon_\perp = \bar{\epsilon}$ we obtain from Eqs. (11) the well known relation $n_{rf} = \sqrt{\bar{\epsilon}\mu}$. Since $\bar{\epsilon} k' \cdot E_0 = 0$ according to Eq. (4) the orientation of the electric field vector E_0 (the polarization) has to be perpendicular to k' but is otherwise undetermined. According to Eqs. (4) the magnetic field vector, which is also perpendicular to k', has to be perpendicular to E_0. For finite $k'_\parallel$ we speak of TM (transverse magnetic) waves in our geometry when the electric field is in the incidence plane spanned by $k', \hat{z}$; i.e., the magnetic field is perpendicular to that plane. Alternatively, for the TE (transverse electric) waves the directions of the electric and magnetic fields are interchanged. For $k'_\parallel = 0$ the TM waves are defined to be polarized along the $\hat{x}$ axis and the TE waves along the $\hat{y}$ axis. Incoming waves with arbitrary polarization obviously correspond to a certain linear superposition of TM and TE waves.

In uniaxial nematic layers n_{rf} depends, in general, on the *ordinary* refraction index, $n_o = \sqrt{\epsilon_\perp \mu}$, and the *extraordinary* one, $n_e = \sqrt{\epsilon_\parallel \mu}$. In addition, the angle between $\hat{\mathbf{n}}$ and k' comes into play. In view of the condition $k' \cdot D_0 = k' \cdot (\boldsymbol{\varepsilon} \cdot E_0) = 0$ [see Eq. (10)] the linear equation $M \cdot E_0 = 0$ [see Eq. (11)] allows for the two linear independent solutions $E_0 = o_0[k'_o]$ (*ordinary*) and $E_0 = e_0[k'_e]$ (*extraordinary*), which are represented as the following unit vectors:

$$o_0[k'_o] = \frac{k'_o \times \hat{\mathbf{n}}}{|k'_o \times \hat{\mathbf{n}}|}, \quad e_0[k'_e] = \frac{\epsilon_\perp \hat{\mathbf{n}} - k'_e(k'_e \cdot \hat{\mathbf{n}})}{|\epsilon_\perp \hat{\mathbf{n}} - k'_e(k'_e \cdot \hat{\mathbf{n}})|}. \quad (12)$$

The parallel components of the wave vectors k'_e and k'_o are given by $k'_\parallel$ of the incident plane wave, as discussed before. Their z components are determined by the refraction indices

$n_{\text{ord}}, n_{\text{ext}}$ given as follows:

$$M[k'_o] \cdot o_0[k'_o] = 0 \implies (k'_o)^2 = n_{\text{ord}}^2 = n_o^2,$$
$$M[k'_e] \cdot e_0(k'_e) = 0 \implies (k'_e)^2 = n_{\text{ext}}^2 = n_e^2 - \beta(k'_e \cdot \hat{\mathbf{n}})^2 = 0,$$
$$\text{with } \beta = \frac{n_e^2}{n_o^2} - 1. \quad (13)$$

The corresponding magnetic field vectors are easy to calculate, since they are perpendicular to $e_0[k'_e], k'_e$ and to $o_0[k'_o], k'_o$, respectively [see Eq. (4)].

To determine the amplitudes of the diffracted and reflected plane waves in dependence on the wave vector and the polarization of the incident plane wave one makes use of the continuity of the tangential components of the electric and magnetic fields at any interface. In addition, the normal components of the displacement D and the magnetic induction B have to be continuous as well. The calculational details are presented in Appendix A.

III. THE SHORT-WAVELENGTH APPROXIMATION FOR THE OPTICS OF MATERIALS WITH UNIAXIAL SYMMETRY

As demonstrated in Sec. II B the general monochromatic Maxwell equations (7) and (8) can easily be solved in terms of plane waves for a constant dielectric tensor $\boldsymbol{\varepsilon}$. We are now interested in the case where $\boldsymbol{\varepsilon}$ varies slowly in space on a scale $2\pi/q$ which is much larger than the wavelength $\lambda = 2\pi/k_0, k_0 = \omega/c$ of the light wave in vacuum. The position dependence of $\boldsymbol{\varepsilon}$ is induced by the spatial variations of $\hat{\mathbf{n}}$.

As described, for instance, in Ref. [17] (Chap. III), it is appropriate to solve the monochromatic Maxwell equations [Eqs. (10)] by using the ansatz

$$E(r) = E_0(r)e^{ik_0 S(r)}, \quad H(r) = H_0(r)e^{ik_0 S(r)}. \quad (14)$$

The *eikonal S* is a real scalar function of position, while the field amplitudes E_0, H_0 are, in general, complex vector functions of position. The ansatz for $E(r)$ in Eq. (14) is now inserted into Eq. (7) where we make use of the general identity:

$$\nabla \times \{\nabla \times (V \exp[ik_0 S])\}$$
$$= \exp[ik_0 S] (\nabla + ik_0 \nabla S)[(\nabla + ik_0 \nabla S) \times V] \quad (15)$$

for an arbitrary vector V and a scalar S, which are both space dependent. Collecting the terms $O(k_0^2)$ we arrive at the eikonal equation:

$$\{(\nabla S)^2 - \nabla S \otimes \nabla S - \mu[\epsilon_\perp + (\epsilon_\parallel - \epsilon_\perp)\hat{\mathbf{n}} \otimes \hat{\mathbf{n}}]\} \cdot E_0 = 0. \quad (16)$$

Comparison with Eq. (11) shows that ∇S corresponds to a local version of the (dimensionless) wave vector k' in the case of constant $\hat{\mathbf{n}}$. Accordingly, we introduce local representations o^N and e^N of the ordinary and extraordinary polarization vectors o_0 and e_0 defined in Eq. (12),

$$o^N(r) = \frac{\nabla S_o(r) \times \hat{\mathbf{n}}(r)}{|\nabla S_o(r) \times \hat{\mathbf{n}}(r)|},$$
$$e^N(r) = \frac{\epsilon_\perp \hat{\mathbf{n}}(r) - \nabla S_e(r)[\nabla S_e(r) \cdot \hat{\mathbf{n}}(r)]}{|\epsilon_\perp \hat{\mathbf{n}}(r) - \nabla S_e(r)[\nabla S_e(r) \cdot \hat{\mathbf{n}}(r)]|}, \quad (17)$$

the position dependence of which is often suppressed in the following. Thus, we have to distinguish between the ordinary eikonal solution, S_o, and the extraordinary one, S_e, of Eq. (16), which correspond to the solutions $\boldsymbol{E}_0 = \boldsymbol{o}^N$ and $\boldsymbol{E}_0 = \boldsymbol{e}^N$ in Eq. (17). Obviously, the expressions for $\boldsymbol{k}'_e$, $\boldsymbol{k}'_o$ in Eq. (13) translate immediately into the following equations for the local wave vectors $\boldsymbol{\nabla} S_o$ and $\boldsymbol{\nabla} S_e$:

$$(\boldsymbol{\nabla} S_o)^2 = n_o^2, \tag{18a}$$

$$(\boldsymbol{\nabla} S_e)^2 + \beta(\boldsymbol{S}_e \cdot \hat{\boldsymbol{n}})^2 = n_e^2. \tag{18b}$$

The solutions S_e^0, S_o^0 of Eqs. (18) for the homogeneous basic planar state where $\hat{\boldsymbol{n}} = \hat{\boldsymbol{n}}_0 = \hat{\boldsymbol{x}}$, which are associated with a plane wave with wave vector $\boldsymbol{k}' = (k'_x, k'_y, k'_z) \equiv \boldsymbol{k}'_\parallel + k'_z \boldsymbol{e}_z$, read as follows:

$$S_o^0(\boldsymbol{x},z) = \boldsymbol{k}'_\parallel \cdot \boldsymbol{x} + k_z'^o(\boldsymbol{k}') z, \quad S_e^0(\boldsymbol{x},z) = \boldsymbol{k}'_\parallel \cdot \boldsymbol{x} + k_z'^e(\boldsymbol{k}') z, \tag{19}$$

with

$$k_z'^o(\boldsymbol{k}') = n_o \left[1 - \frac{k_x'^2 + k_y'^2}{n_o^2} \right]^{1/2},$$

$$k_z'^e(\boldsymbol{k}') = n_e \left[1 - \frac{k_x'^2}{n_o^2} - \frac{k_y'^2}{n_e^2} \right]^{1/2}. \tag{20}$$

It is convenient to split off from the eikonal S_e in Eq. (18b) the contribution S_e^0 in Eq. (19) of the basic state by using the ansatz $S_e = S_e^0 + \bar{S}_e$. Thus, one arrives at a modified eikonal equation for $\bar{S}_e$ as a quadratic form in the partial derivatives $\partial_z \bar{S}_e$:

$$A \left(\frac{\partial}{\partial z} \bar{S}_e \right)^2 + 2B \left(\frac{\partial}{\partial z} \bar{S}_e \right) - C = 0, \quad \bar{S}_e = S_e - S_e^0. \tag{21}$$

The coefficients A, B, and C depend on the derivatives $\partial_x \bar{S}_e, \partial_y \bar{S}_e$ and on the director distortion $\Delta \boldsymbol{n}$. In analogy to [18], Eq. (21) can be rewritten as

$$\frac{\partial}{\partial z} \bar{S}_e = \frac{-B + \sqrt{B^2 + AC}}{A} = \frac{C}{B + \sqrt{B^2 + AC}}. \tag{22}$$

Equation (22) has to be solved with the initial condition $\bar{S}_e(\boldsymbol{x}, z = 0) = 0$, since $S_e(\boldsymbol{x}, 0) \equiv S_e^0(\boldsymbol{x}, 0)$, which is already split off in Eq. (21). Note that for the same reason we had to exclude the negative square root in the transition from Eqs. (21) to (22).

For the director distortion $\Delta \boldsymbol{n}$ given in Eq. (1) we restrict ourselves to solutions of Eq. (22) up to second order in the small quantities θ_m and $|\boldsymbol{k}'_\parallel|$, i.e., allowing only for small deviations from perpendicular incidence ($|\boldsymbol{k}'_\parallel| = 0$). Thus, we represent $\bar{S}_e$ in the form

$$\bar{S}_e(\boldsymbol{x},z) = \theta_m S^{(1,0,0)}(\boldsymbol{x},z) + k'_x \theta_m S^{(1,1,0)}(\boldsymbol{x},z)$$
$$+ k'_y \theta_n S^{(1,0,1)}(\boldsymbol{x},z) + \theta_m^2 \bar{S}^{(2)}(\boldsymbol{x},z) + \cdots. \tag{23}$$

Note that θ_m-independent terms do not appear; they are already covered by $S_e^0(\boldsymbol{x},z)$ [see Eq. (19)]. To proceed it is sufficient to keep in the coefficients A, B, C of Eq. (22) only the terms contributing to the expansions coefficients of $\bar{S}_e$ in Eq. (23). After simple algebra we thus arrive at the following leading terms which depend on the components of the director

distortion $\Delta \boldsymbol{n}$ [see Eq. (1)] and on the components of $\boldsymbol{k}'$ as follows:

$$A = 1, \quad B = k_z'^e + \beta \theta_m \delta n_z k'_x,$$
$$C = -\beta n_e^2 \theta_m^2 (\delta n_z)^2 - 2\beta k'_x n_e \delta n_z. \tag{24}$$

Using these expressions we obtain the following ordinary differential equations for $S^{(1,1,0)}(\boldsymbol{x},z)$ and $S^{(2)}(\boldsymbol{x},z)$:

$$\frac{\partial S^{(1,1,0)}(\boldsymbol{x},z)}{\partial z} = -\beta \delta n_z(\boldsymbol{x},z),$$
$$\frac{\partial S^{(2)}(\boldsymbol{x},z)}{\partial z} = -\frac{\beta}{2} n_e (\delta n_z(\boldsymbol{x},z))^2, \tag{25}$$

while $S^{(1,0,0)} \equiv 0$ and $S^{(1,0,1)} \equiv 0$. Note, that neither a twist of the director in the plane ($\delta n_y \neq 0$) nor finite values of k'_y are reflected in $\bar{S}_e$ in this order. According to Eq. (25) the calculation of the eikonal solution requires only z integrations. Choosing for $\delta n_z(\boldsymbol{x},z)$ in particular the one-mode representation in Eq. (3) we obtain in this way

$$\bar{S}_e(\boldsymbol{x},z) = -\theta_m d \frac{\beta}{4\pi} \cos(\boldsymbol{q} \cdot \boldsymbol{x}) \left\{ 8k'_x \sin^2(\pi z/2d) \right.$$
$$\left. + \theta_m \frac{n_e}{2} \left[\frac{2z\pi}{d} - \sin(2\pi z/d) \right] \cos(\boldsymbol{q} \cdot \boldsymbol{x}) \right\}. \tag{26}$$

In Appendix E we discuss the general solution of Eq. (22) up to orders $O(\theta_m^2)$ and $|\boldsymbol{k}'_\parallel|^2$, where the impact of finite k'_y and δn_y becomes visible.

In a next step we have to determine the amplitude $\boldsymbol{E}_0$ [Eq. (14)] of the electric field, which is expanded in terms of $\boldsymbol{o}^N$ and $\boldsymbol{e}^N$ and a third linearly independent vector, $\boldsymbol{s}^N$. Following [18], $\boldsymbol{s}^N$ is defined as

$$\boldsymbol{s}^N = \frac{\boldsymbol{\nabla}(S_o + S_e)}{|\boldsymbol{\nabla}(S_o + S_e)|}; \quad \text{thus, } \boldsymbol{s}^N = \hat{\boldsymbol{z}} \text{ for } \boldsymbol{k}'_\parallel = 0, \tag{27}$$

and the electric field is finally represented as

$$\boldsymbol{E}(\boldsymbol{r}) = \mathbb{O}(\boldsymbol{r}) \boldsymbol{o}^N(\boldsymbol{r}) e^{i k_0 S_o} + \mathbb{E}(\boldsymbol{r}) \boldsymbol{e}^N(\boldsymbol{r}) e^{i k_0 S_e}$$
$$+ i \mathbb{Z}(\boldsymbol{r}) \frac{\boldsymbol{s}^N(\boldsymbol{r})}{k_0} e^{i k_0 (S_e + S_o)/2}, \tag{28}$$

with amplitudes $\mathbb{O}$, $\mathbb{E}$, and $\mathbb{Z}$ and the eikonal solutions $S_o(\boldsymbol{r}), S_e(\boldsymbol{r})$ of Eqs. (18). Introducing the ansatz Eq. (28) into Eq. (7) and making use of Eq. (15) leads eventually to coupled partial differential equations for $\mathbb{O}(\boldsymbol{x},z), \mathbb{E}(\boldsymbol{x},z)$ presented in Appendix B. The equations are solved iteratively as power series in θ_m, where the power series expansion of the extraordinary eikonal $\bar{S}_e(\boldsymbol{x},z)$ [see Eq. (26)] serves as input. Some explicit results will be given in Secs. IV and V and discussed in Sec. VI. In the following section, however, it will be demonstrated that the determination of the amplitudes $\mathbb{O}, \mathbb{E}$ can be circumvented in the frequently used two-dimensional experimental geometry.

It should be emphasized that in the approach used in Ref. [10] the steps of our procedure are just reversed. First the field amplitudes are determined by using Fermat's principle to determine the light ray trajectories, which is technically cumbersome. Then the eikonal S_e is obtained by summing up the phases along the ray path.

IV. THE SHORT-WAVELENGTH APPROXIMATION IN A 2D GEOMETRY

In this section we restrict ourselves to the special case of an incident TM wave, where the wave vector $\boldsymbol{k}'$ and the electric field amplitude $\boldsymbol{E}_0$ of the plane wave propagating into the nematic layer are confined to the xz plane. Furthermore, the director distortion $\delta\boldsymbol{n}$ defined in Eq. (3) is assumed to be parallel to this plane as well. Finally, we consider only stripe pattern (called "normal" rolls in EC), where the wave vector $\boldsymbol{q}$ characterizing $\delta\boldsymbol{n}$ in Eq. (1) is parallel to $\boldsymbol{n}_0 = \hat{\boldsymbol{x}}$, the director orientation in the basic state. Thus, all quantities depend only on x, z. Inspection of the Maxwell equations reveals that $\mathbb{E}$ remains in the xz plane inside the nematic layer, i.e., $\mathbb{O}(x, z) \equiv 0$. Only this exclusively extraordinary configuration has been discussed in the literature [7,10,15] so far.

Instead of analyzing equations for the electric field [see Eqs. (B3) in Appendix B] we approach the problem from a slightly different perspective. In the 2D geometry the magnetic induction $\boldsymbol{B}$ has only a nonvanishing y component, B_y, and it is expected that an analysis based on the y component of Eq. (8), a scalar equation, is more transparent and straightforward than the analysis based on the electric field. First of all, we use the fact that in the present geometry Eq. (8) leads to the following scalar equation for $B_y(x, z)$:

$$\nabla \cdot (\boldsymbol{\varepsilon} \cdot \nabla B_y) + \epsilon_{\parallel}\epsilon_{\perp}\mu k_0^2 B_y = 0. \quad (29)$$

Instead of solving Eq. (29) by using a representation of B_y with a complex phase as in Ref. [16] we follow closely the approach discussed in Sec. III using the ansatz

$$B_y(x, z) = \bar{B}(x, z)\exp[ik_0 S_e(x, z)], \quad (30)$$

which requires much less effort. We arrive again at the eikonal equation (18b) for S_e discussed before in Sec. III and to the following equation for $\bar{B}(x, z)$:

$$\nabla S \cdot [\boldsymbol{\varepsilon} \cdot \nabla \bar{B}(x, z)] + \nabla[\bar{B}(x, z)(\boldsymbol{\varepsilon} \cdot \nabla \bar{S}_e)] = 0, \quad (31)$$

with $\bar{S}_e = S_e - S_e^0$ [see Eq. (21)].

We concentrate on the case of small θ_m and small k'_x and use the following expansion for the magnetic field amplitude $\bar{B}(x, z)$ (31):

$$\bar{B}(x, z) = B_0 + \theta_m B^{(1)}(x, z) + \theta_m^2 B^{(2)}(x, z)$$
$$+ \theta_m k'_x B^{(1,1)}(x, z). \quad (32)$$

To order θ_m Eq. (31) reduces to

$$2\frac{\partial}{\partial z}B^{(1)}(x, z) = -\beta B_0 \frac{\partial}{\partial x}\delta n_z(x, z), \quad (33)$$

which has to be solved with the initial condition $B^{(1)}(x, z = 0) = 0$. When using again the one-mode approximation for $\delta n_z(x, z)$ in Eq. (3) we arrive by direct integration at

$$B^{(1)}(x, z) = B_0 qd\frac{\beta}{2\pi}\sin(qx)[1 - \cos(\pi z/d)]. \quad (34)$$

The higher order amplitudes defined in Eq. (32), which give also contributions to the amplitude-grating efficiency of the distorted nematic layer, are given in Appendix C. For small director distortion amplitudes θ_m, considered in this work, they are much smaller than the corresponding phase-grating terms

$\propto k_0\bar{S}_e(x, z)$ [see Eq. (26)]. Thus, in the following only the leading amplitude term $\propto\theta_m$ like $B^{(1)}$ will be kept.

The expansion coefficients of the total magnetic field amplitude $\bar{B}(x, d)/B_0$ defined in Eq. (32) agree with those given in Ref. [16]. Furthermore, they allow for the calculation of the electric field by using the general Maxwell equations (4). In fact, we arrive in this way at the electric field amplitudes $\mathbb{E}(x, d)/\mathbb{E}_0$ [see Eq. (28)], as calculated in Ref. [10] by considerably more intricate calculations. For completeness we have convinced ourselves that also the direct solution of Eqs. (B3) leads to the same result. This perfect agreement (for more details, see Ref. [25]) serves as a most convincing test of our procedure.

V. OPTICAL ANALYSIS OF GENERAL 3D PATTERN

In the following we describe the optical properties of general 3D configurations in nematics. First, the wave vector $\boldsymbol{q}$ characterizing the periodic distortion $\delta\boldsymbol{n}$ [see Eq. (3)] may not be parallel to $\boldsymbol{n}_0 = \hat{\boldsymbol{x}}$. Furthermore, both the electric field polarization and the orientation of the wave vector $\boldsymbol{k} = k_0\boldsymbol{k}'$ of the incident plane wave are, in principle, arbitrary. In general, we use the following representation for the (dimensionless) wave vector $\boldsymbol{k}'$ of the plane wave, which enters the nematic layer from a glass plate (refraction index n_g):

$$\boldsymbol{k}' = \boldsymbol{k}'_{\parallel} + k'_z\hat{\boldsymbol{z}}$$
$$= \left[\sin(\vartheta_g)\cos\phi,\ \sin(\vartheta_g)\sin\phi,\ \sqrt{n_g^2 - \sin^2(\vartheta_g)}\right]. \quad (35)$$

Thus, $|\boldsymbol{k}'|^2 = n_g^2$ (see Sec. II B) is automatically guaranteed. The polar angle ϑ_g describes the inclination of the incoming ray with respect to the layer normal (parallel to $\hat{\boldsymbol{z}}$) and the azimuthal angle, ϕ, a rotation of $\boldsymbol{k}'$ about this axis. It is obvious that the angles ϑ_g and ϕ can be alternatively interpreted in terms of a tilt and a rotation of the nematic layer at fixed $\boldsymbol{k}'$. In line with the typical experimental setup we restrict ourselves in the following to small $|\boldsymbol{k}'_{\parallel}|$, i.e., to small ϑ_g. Then up to order $O(\sin(\vartheta_g))$ the wave vectors of the extraordinary and the ordinary waves inside the nematic layer are given as $\boldsymbol{k}' = (\boldsymbol{k}'_{\parallel}, n_e)$ and $\boldsymbol{k}' = (\boldsymbol{k}'_{\parallel}, n_o)$, respectively (see Sec. II B).

Let us start with a "toy model" of a periodic twist modulation of the director confined to the xy plane of the form

$$\hat{\boldsymbol{n}} = (\cos\alpha(x, z), \sin\alpha(x, z), 0), \quad (36)$$

with the twist angle $\alpha(x, z) = \alpha_m\vartheta(z)\cos(qx)$, where $\vartheta(z) = 0$ for $z = 0, d$. In the absence of x variations this director configuration is realized in the planar geometry as the result of a twist Freedericksz transition, when a static magnetic field with amplitude H above a critical threshold field H_c is applied along the y direction. The maximal twist amplitude α_m varies then like $(H^2 - H_c^2)^{1/2}$. An additional x variation is, for instance, characteristic for the so called chevron patterns in EC [26], when the applied ac voltage is turned off after some time.

We consider the special case of an incident TM wave in the xz plane, which leads to an extraordinary wave with amplitude $\mathbb{E}_0$ at the lower surface $z = 0$ of the nematic layer [see Eq. (28)]. At $z = d$ we find a TM wave as well, since the polarization of a plane wave follows adiabatically (Mauguin

principle) the orientation of $\boldsymbol{e}^N(x,z)$ (17) and is thus again parallel to $\hat{\boldsymbol{x}}$ at $z=d$. Due to the twist in the director field, however, a TE electric field component is observable as well in the upper glass plate, since also an ordinary wave with amplitude $\mathbb{O}(x,z)$ develops in the layer. To leading order in α_m, k_x we obtain easily from Eqs. (B3) the following partial differential equation for $\mathbb{O}(x,z)$:

$$\left(n_o \frac{\partial}{\partial z} + k_x \frac{\partial}{\partial x} \right) \mathbb{O}(x,z)$$
$$= e^{i k_0 (n_e - n_o)z} \mathbb{E}_0 \left(n_e \frac{\partial}{\partial z} + k_x \frac{\partial}{\partial x} \right) \cos(qx) \vartheta(z). \quad (37)$$

The solution of Eq. (37) in Fourier space by using the ansatz

$$\mathbb{O}(x,z) = \mathbb{O}'(q,z) \exp[i\, qx] + \mathbb{O}'(-q,z) \exp[-i\, qx] \quad (38)$$

reads as follows:

$$\mathbb{O}'(x,\pm q) = \mathbb{E}_0 \alpha_{\max} \frac{n_e}{2n_o} \exp[\pm i\, qk_x'z]$$
$$\times \int_0^z dz' \exp[i\, k_0(n_e - n_o \mp k_x'q)z]$$
$$\times (\partial_{z'} \pm i k_x'q) \vartheta(z'). \quad (39)$$

For perpendicular incidence ($k_x' = 0$), we obtain thus

$$\mathbb{O}(x,z) = \mathbb{E}_0 \alpha_m \frac{n_e}{n_o} \cos(qx) \int_0^z dz' e^{i\, k_0(n_e - n_o)z'} \frac{d}{dz'} \vartheta(z'). \quad (40)$$

This expression was first derived in Ref. [27] for $q=0$ by a different method. There it has been emphasized that the electric field intensity $|\mathbb{O}(z=d)|^2$, which is singled out and measured by using crossed polarizers at $z=0,d$, gives valuable information on the elastic constants of nematics. Note also the sensitive dependence of $|\mathbb{O}(z=d)|^2$ on ω, d via the large exponent $k_0 d(n_e - n_o)$. It should be mentioned that the amplitude $\mathbb{O}(x,z)$ in Eq. (40) has been also derived in Ref. [28] by using the Jones matrix method, except that the prefactor n_e/n_o is approximated by one.

After these introductory considerations we now deal with the general and mostly realized case of director distortions which contain also a tilt contribution [see Eq. (3)]. The corresponding eikonals, S_o, S_e, have been already given in Sec. III [see Eqs. (19) and (26)]. To determine the field amplitudes $\mathbb{E}(\boldsymbol{x},z), \mathbb{O}(\boldsymbol{x},z)$ defined in Eq. (28) we have to solve Eqs. (B3). We restrict ourselves to solutions up to order θ_m and may neglect, as discussed in Appendix C, the dependence of the amplitudes on $|\boldsymbol{k}_\parallel'|$. Thus, the field amplitudes are expanded as follows:

$$\mathbb{O} = \mathbb{O}_0 + \theta_m \mathbb{O}^{(1)}(\boldsymbol{x},z), \quad \mathbb{E} = \mathbb{E}_0 + \theta_m \mathbb{E}^{(1)}(\boldsymbol{x},z). \quad (41)$$

To first order in θ_m we arrive from Eqs. (B3) easily at the following partial differential equations for $\mathbb{O}^{(1)}, \mathbb{E}^{(1)}$ in terms of the components of $\Delta \boldsymbol{n}$ [see Eq. (1)]:

$$\frac{\partial}{\partial z} \mathbb{E}^{(1)}(\boldsymbol{x},z) = -\mathbb{E}_0 \frac{\beta}{2} \frac{\partial}{\partial x} \delta n_z(\boldsymbol{x},z) - \mathbb{O}_0 \frac{a_-^2 n_o}{n_e} \frac{\partial}{\partial z} \delta n_y(\boldsymbol{x},z), \quad (42a)$$

$$\frac{\partial}{\partial z} \mathbb{O}^{(1)}(\boldsymbol{x},z) = \mathbb{E}_0 \frac{a_+^2 n_e}{n_o} \left[\frac{\partial}{\partial z} \delta n_y(\boldsymbol{x},z) + \frac{\beta}{2} \frac{\partial}{\partial y} \delta n_z(\boldsymbol{x},z) \right]. \quad (42b)$$

According to Eqs. (B3) the functions $a_\pm^2(\boldsymbol{x},z)$ are given as

$$a_\pm^2(\boldsymbol{x},z) = \exp[\pm i\, k_0((n_e - n_o)z + \bar{S}_e(\boldsymbol{x},z))], \quad (43)$$

where $\bar{S}_e$ within the one-mode approximation can be found in Eq. (26). Inspection of the transmission coefficients in Eq. (A4) shows that the initial ordinary and extraordinary electric field amplitudes $\mathbb{O}_0, \mathbb{E}_0$ at $z=0$ can be realized by using the incident electric field as a superposition of a TM wave and a TE wave.

Equations (42) are easy to solve and we obtain

$$\mathbb{E}^{(1)}(\boldsymbol{x},z) = \mathbb{E}_0 \left[1 - \frac{\beta}{2} \int_0^z dz' \frac{\partial}{\partial x} \delta n_z(\boldsymbol{x},z') \right]$$
$$- \mathbb{O}_0 \frac{n_o}{n_e} \int_0^z dz' a_-^2(\boldsymbol{x},z') \frac{\partial}{\partial z} \delta n_y(\boldsymbol{x},z'), \quad (44a)$$

$$\mathbb{O}^{(1)}(\boldsymbol{x},z) = \mathbb{O}_0 + \mathbb{E}_0 \frac{n_e}{n_o} \int_0^z dz' a_+^2(\boldsymbol{x},z')$$
$$\times \left[\frac{\partial}{\partial z} \delta n_y(\boldsymbol{x},z') + \frac{\beta}{2} \frac{\partial}{\partial y} \delta n_z(\boldsymbol{x},z') \right]. \quad (44b)$$

The final expressions in Eqs. (44) are not difficult to interpret. First of all, the extraordinary field amplitude $\mathbb{E}^{(1)}$, obtained before in the 2D geometry [see Eq. (33)], where $\mathbb{O}_0 = 0$ and $\boldsymbol{q} = (q,0)$, is recovered, as it should be. Of particular importance is the fact that according to Eq. (44a) a finite incident ordinary field amplitude $\mathbb{O}_0$ at $z=0$ is sufficient to generate an extraordinary field component for finite z and thus also at $z=d$ [9]. Analogously, we obtain in Eq. (44b) from a nonzero $\mathbb{E}_0$ a finite ordinary field amplitude $\mathbb{O}(\boldsymbol{x},d)$. In close analogy to Eq. (40) one needs in any case a twist of the director field ($\delta n_y \neq 0$) or a finite angle between the wave vector $\boldsymbol{q}$ and $\boldsymbol{n}_0$ [leading to a y dependence of δn_z; see Eq. (3)]. Note that given the amplitudes $\mathbb{E}^{(1)}$ and $\mathbb{O}^{(1)}$ the full extraordinary and ordinary electric fields are obtained according to Eq. (28) by a multiplication with the corresponding phase factors and the polarization vectors. The detailed discussion is postponed to Sec. VI.

We have demonstrated in this section and before in Sec. IV that the electromagnetic field at the upper surface $z=d$ of the nematic layer shows periodic variations with wave vector $\boldsymbol{q}$ both in the amplitude (amplitude grating) and in the the phase (phase grating). Using these data the field in the isotropic media (glass, air) above the nematic layer has to be constructed by solving the isotropic Maxwell equations with constant dielectric constants. This task is formally easy by expanding the fields at $z=d$ into Fourier series in terms of $e^{i\, n\boldsymbol{q}\cdot\boldsymbol{x}}, n=\pm 1, \pm 2 \ldots$; i.e., the field is represented as a superposition of plane waves. The whole procedure and its implications have been discussed in Ref. [10] for RBC. The corresponding analysis for nematics is presented in Appendix D. The moduli of the Fourier components give directly the intensity of the diffraction fringes. The interference of the plane waves produces the shadowgraph picture. The main results are collected and discussed in the following section.

VI. SUMMARY AND DISCUSSION

In the previous sections we have analyzed the optical properties of a thin nematic layer of thickness d in the presence

of small director perturbations $\Delta \boldsymbol{n}(\boldsymbol{x},z)$ [Eq. (1)] of the planar basic state ($\boldsymbol{n}_0 = \hat{\boldsymbol{x}}, 0 \leqslant z \leqslant d$). The perturbation is periodic in the xy plane with wave vector $\boldsymbol{q}$. The optical properties of the layer are probed using a monochromatic plane wave with frequency ω and wave vector $\boldsymbol{k}$ [see Eq. (35)], which enters the layer at $z = 0$ from a glass plate with refraction index n_g. The goal of this section is to summarize and discuss the main results on the optical analysis, which have been described in Secs. IV and V, as well as in several Appendixes.

The incident wave can, in general, be decomposed into a superposition of a TE and TM wave with polarization vectors $\boldsymbol{p}_E, \boldsymbol{p}_M$ defined in Appendix A. Using the transfer coefficients given there the amplitudes $\mathbb{O}_0$ and $\mathbb{E}_0$ of the ordinary and extraordinary waves just at the lower surface at $z = 0$ of the nematic layer are directly given. Inside the nematic layer ($0 \leqslant z \leqslant d$) the electric field is represented by an amplitude and a phase factor as discussed in Sec. III. Let us first concentrate on the extraordinary waves which are almost exclusively utilized in EC experiments, where $\boldsymbol{q}$ includes, at most, a moderate angle with $\boldsymbol{n}_0 \parallel \hat{\boldsymbol{x}}$. Furthermore, only the leading terms in inclination angle ϑ_g [Eq. (35)] of the incident plane wave are kept. In this approximation the extraordinary electric field $\boldsymbol{E}_e(\boldsymbol{x})$ at $z = d$ given in Eq. (D1) reads as

$$\boldsymbol{E}_e(\boldsymbol{x}) = \mathbb{E}_0 \exp[i\, k_0(\boldsymbol{k}'_\parallel \cdot \boldsymbol{x} + n_e d)]$$
$$\times \sum_{n=-\infty}^{\infty} C^N(n) e^{i\, n(\boldsymbol{q} \cdot \boldsymbol{x})} \boldsymbol{e}_0[\boldsymbol{k}'_\parallel]. \quad (45)$$

The Fourier coefficients $C^N(n)$ describe the optical-grating characteristics of the nematic fluid layer at $z = d$ and eventually also the intensity of the TM wave in air above the nematic layer [see Eq. (D12)].

The intensity of the two diffraction spots of order n relative to the intensity $|\mathbb{E}_0 T_{eg} T_{ga}|^2$ of the transmitted wave [see Eq. (D12)] is, in general, given as $|C^N(\boldsymbol{n})|^2$. The diffraction spots characterized by the wave vectors $\boldsymbol{k} = (k_0 k' \pm nq)\hat{\boldsymbol{x}} + k_0 \hat{\boldsymbol{z}}$ are observed in directions which enclose small angles $\gamma_{\pm n}$ with $\hat{\boldsymbol{z}}$ given by

$$\gamma_{\pm n} = \arctan\left[\frac{|k_0 \boldsymbol{k}'_\parallel \pm \boldsymbol{n}q|}{k_0}\right], \quad (46)$$

where we have neglected the terms of the order $O(n^2 q^2 / k_0^2)$.

Let us start with the contribution of the leading Fourier coefficients $C^N(\pm 1)$. The relative intensities of the diffraction spots are determined by $|C^N(\pm 1)|^2$, which, according to Eq. (D4), are given as follows:

$$|C^N(\pm 1)|^2 = \frac{\theta_m^2}{4} |c_{E1}|^2 \left|1 \mp \frac{c_{S1}}{c_{E1}}\right|^2, \quad \text{with}$$
$$\frac{c_{S1}}{c_{E1}} = -\frac{2k_0}{q_x} \sin(\vartheta_g) \cos\phi. \quad (47)$$

Here we have inserted the expressions for the coefficients c_{S1}, c_{E1} in Eq. (D3).

At first a small asymmetry between $n = 1$ and $n = -1$ in the case of oblique incidence ($\vartheta_g \neq 0$) is evident. The asymmetry is reflected both in the angles $\gamma_{\pm 1}$ [Eq. (46)] and in the intensities [Eq. (47)]. A closer look at the intensities reveals a competition between the amplitude-grating (c_{E1}) and phase-grating coefficients (c_{S1}). For $\phi = 0$ already at the small

inclination angle $\vartheta_g = q_x / (2k_0)$ phase grating starts to prevail. For standard EC experiments and medium ac frequencies Ω where typically $qd \approx 1.5\pi$ we obtain thus $\vartheta_g \approx 0.012 \approx 0.7°$ when using visible light with $\lambda = 0.6$ μm and a layer thickness of 20 μm. Consequently, for an angle of $7°$, say, the intensity $|C^N(1)|^2$ is larger by a factor of 100 compared to perpendicular incidence ($\vartheta_g = 0$). For smaller Ω in EC often oblique rolls appear; the angle between $\boldsymbol{q}$ and $\boldsymbol{n}_0$ becomes finite, while the modulus of $\boldsymbol{q}$ remains practically unchanged. Since thus the x component, q_x, of the wave vector $\boldsymbol{q}$ decreases, the denominator of the ratio c_{S1}/c_{E1} [Eq. (47)] gets larger and phase grating becomes dominant at even smaller ϑ_g. On the other hand, an azimuthal rotation of the incidence plane of the incident light (ϕ finite) leads to a reduction of the phase-grating effect due to the factor $\cos\phi$ in the ratio c_{S1}/c_{E1}. In conclusion, we have demonstrated that even a small inclination of the incoming ray leads to a dramatic increase of the intensity of the first order fringes. So far we are only aware of one systematic investigation of this effect [15], where the main experimental findings have been well described exclusively on the basis of phase grating ($c_{E1} = 0$). For completeness, we refer to a later work [29], where a small inclination of the nematic has been applied as well to enhance the intensity of the first order fringe.

Let us now consider the intensity of the second order fringes ($n = \pm 2$), in particular in relation to the intensity of the first order fringes. In the case of perpendicular incidence of the light and $\phi = 0$ we have thus to compare $|C^N(2)| = \theta_m^2 n_e k_0 d\beta / 16$ [see Eq. (D4)] with $|C^N(1)| = \theta_m q_x d\beta / (2\pi)$. Using as in the previous paragraph the ratio $q_x / k_0 = 1/40$ we find that for $\theta_m > \theta_{m2} = 8q_x / (k_0 n_e \pi) \approx 0.064/n_e$ the intensities of the second order fringes start to outcompete the first order ones. For MBBA where $n_e = 1.75$ one has $\theta_{m2} = 2°$. Using the relation $\theta_m = 80° \sqrt{\epsilon}$ between the maximal tilt angle θ_m and the control parameter $\epsilon = (U_0^2 - U_c^2)/U_c^2$ in MBBA mentioned at the end of Sec. II A the value of $\theta_{m2} = 2°$ corresponds to $\epsilon = 0.0007$. It means that already not too far from threshold the the intensity of the second order fringe becomes considerably larger than the intensity of the first order fringe. This general trend has been clearly demonstrated in experiments [15], where the first order fringes are not visible at all in the case of perpendicular incidence. When inclining the layer, however, we have to compare $\theta_m c_{S1}/2$ with $\theta_m^2 c_{S2}/4$, i.e., $(k_0 d)\beta \sin(\vartheta_g)/\pi$ with $\theta_m(k_0 d) n_e \beta / 16$. Thus, for $\vartheta_g > \theta_m n_e \pi / 16$ the first order fringe prevails. For MBBA this is, for instance, the case for $\vartheta_g > 2°$ at $\epsilon = 0.005$.

Let us now briefly analyze the shadowgraph intensity. First we concentrating on the contribution, I_s^1, which is characterized by the wave vector $\boldsymbol{q}$. According to Appendix D, we obtain the following expression to order $O(\theta_m)$:

$$I_s^1(\boldsymbol{x},z) = (E_0 T_{eg} T_{ga})^2 \left\{ 1 + 2\theta_m \left[c_{S1} \cos(\boldsymbol{q} \cdot \boldsymbol{x}) \sin\left(\frac{q^2}{2k_0} z'\right) \right.\right.$$
$$\left.\left. + c_{E1} \sin(\boldsymbol{q} \cdot \boldsymbol{x}) \cos\left(\frac{q^2}{2k_0} z'\right) \right] \right\}, \quad (48)$$

with $z' = z - (d + d_g)$. At first we observe the well known periodicity of the intensity as a function z characterized by the Talbot wavelength $\lambda_T = 4\pi k_0 / q^2$. For the values $q = 1.5\pi/d$ and $\lambda = 2\pi/k_0 = 0.6$ μm we obtain thus $\lambda_T = 1.1$ mm for

$d = 20\ \mu$m. The Talbot periodicity, in clear contrast to the caustics in geometric optics as discussed in Ref. [10] in detail, can be easily demonstrated by focusing a microscope on different z levels. There exists always a contribution to I_s^1 due to amplitude grating ($\propto c_{E1}$) but as discussed before the contrast can be considerably increased via the phase-grating coefficient ($\propto c_{S1}$) in the case of oblique incidence. The Fourier modes $\propto \exp[\pm 2i\,\boldsymbol{q}\cdot\boldsymbol{x}]$ of the shadowgraph intensity, I_s^2, derive from the terms $O(\theta_m^2)$ in Eq. (D4). As mentioned before the coefficient c_{S2} gives, in general, the dominant contribution. Thus, we find in second order

$$I_s^2(\boldsymbol{x},z) = (E_0 T_{eg} T_{ga})^2 \theta_m^2 c_{S2} \cos(2\boldsymbol{q}\cdot\boldsymbol{x}) \cos\left(\frac{4q^2}{2k_0} z'\right). \quad (49)$$

It has been already stressed before that in the case of perpendicular incidence this terms dominates the Fourier coefficients of I_s^1 with respect to $\boldsymbol{x}$ even at small θ_m, in line with the typical experimental observations.

So far we have demonstrated that the use of obliquely incident light is certainly very important to increase the contrast, since we obtain phase-grating contributions with wave vector $\boldsymbol{q}$. One might ask whether a rotation of the incident plane about the z axis with an angle ϕ might have some additional advantages. It is for sure that we have only to analyze a possible impact on phase grating since amplitude-grating effects are much smaller. For this purpose we have given the eikonal solutions also with respect to the ϕ dependence in Appendix E [see Eqs. (E7) and (E10)]. The only term of the order $O(\sin(\vartheta_g))$ is the term $f_c(d) \propto \cos(\phi)$ in Eq. (E9) [corresponding to c_{S1} in Eq. (D3)], which is maximal for $\phi = 0$. First in order $O(\sin^2(\vartheta_g))$ terms $\propto \sin(2\phi)$ appear. There seems to be no chance to disentangle these terms from the other ones in experiment. The same problem appears in the contributions with wave vector $2\boldsymbol{q}$ [see Eqs. (E11)]. Also here the term $k_0 d\, f_c^{(2)}(d)$ in Eq. (E11a) [corresponding to c_{S2} in Eq. (D3)] dominates the other ones. Thus, a rotation of the incidence plane does not yield any advantage.

Finally, we come to a more detailed discussion of the expressions for electric field amplitudes derived in Eqs. (44), which describe a possible rotation of the polarization vectors. Let us first consider the first term $\propto \mathbb{E}_0$ on the right-hand side of Eq. (44a). The z integral gives a nonzero contribution when the $\boldsymbol{q}$ vector has a finite x component q_x and one recovers in the one-mode approximation for $\delta\boldsymbol{n}$ [Eq. (3)] the coefficient c_{E1} in Eq. (D3) and discussed after Eq. (47). The integrals containing the factors $a_\pm^2(\boldsymbol{x},z)$ in Eq. (44) can be easily evaluated in the one-mode approximation $\vartheta(z) = \sin(\pi z/d)$ by exploiting $k_0 d \gg 1$. We use the identity

$$a_\pm^2(\boldsymbol{x},z) = \left\{\pm i\,k_0\left[n_e - n_0 - \beta\delta n_z^2(\boldsymbol{x},z)/2 - k_x'\beta\delta n_z\right]\right\}^{-1}$$
$$\times \frac{\partial}{\partial z} a_\pm^2(\boldsymbol{x},z), \quad (50)$$

where we have explicitly introduced the z derivatives of the eikonal solution in the exponents of $a_\pm^2$ [Eq. (43)] by using Eqs. (25). Then we rewrite the integral in Eq. (44b) with the help of an integration by parts, where only the term $\propto \partial_z \delta n_y(\boldsymbol{x},z)$ gives a nonzero contribution at the boundaries

$z = 0,d$. Thus, we arrive at

$$\mathbb{E}^{(1)}(\boldsymbol{x},d) = \mathbb{E}_0\left[1 + \frac{\beta}{\pi}(q_x d)\sin(\boldsymbol{q}\cdot\boldsymbol{x})\right] - \mathbb{O}_0\theta_m a_y$$
$$\times \frac{n_o}{n_e}\frac{i\,\pi\{1 + \exp[-i\,k_0\Phi(\boldsymbol{x})]\}}{k_0 d(n_e - n_o)}\sin(\boldsymbol{q}\cdot\boldsymbol{x}),$$
$$\text{with}\quad \Phi(\boldsymbol{x}) = S_e(\boldsymbol{x},d) = (n_e - n_o)d + \bar{S}_e(\boldsymbol{x},d), \quad (51)$$

where $\bar{S}_e(\boldsymbol{x},d)$ is explicitly given in Eq. (26). According to Eq. (28) the amplitude $\mathbb{E}^{(1)}(\boldsymbol{x},d)$ has now to be multiplied by the factor $\exp[ik_0 S_e(\boldsymbol{x},d)]\,\boldsymbol{e}_0[\boldsymbol{k}']$ to obtain the extraordinary electric field. In any case, as long as $\delta n_y \neq 0$, we obtain a small amplitude-grating term $\propto \mathbb{O}_0\theta_m \sin(\boldsymbol{q}\cdot\boldsymbol{x})(k_0 d)^{-1}$, where the polarization vector of the incident wave is rotated by $90°$, when leaving the layer. Even for $q_x = 0$, but nonzero $\boldsymbol{k}_\parallel' \propto \sin(\vartheta_g)$ and $\mathbb{E}_0$, however, we obtain a much larger phase-grating contribution $\propto (k_0 d)\theta_m \sin(\vartheta_g)\cos(\boldsymbol{q}\cdot x)$, corresponding to the coefficient c_{S1} discussed before.

In a similar manner we may discuss the amplitude $\mathbb{O}^{(1)}(\boldsymbol{x},d)$ in Eq. (44b). The integral is evaluated again in the one-mode approximation as before, where only the term $\propto \partial_z \delta n_y$ survives in leading order in $(k_0 d)^{-1}$. The total ordinary field is obtained by multiplication of $\mathbb{O}^{(1)}$ with $\boldsymbol{o}_0$ and with the phase factor $\exp[i\,k_0 S_o^0(\boldsymbol{x},d)]$, where S_o^0 [see Eq. (19)] does not contain periodic phase modulation terms. Thus, only the term $\propto \mathbb{E}_0$ leads to optical-grating effects and we obtain the total ordinary electric field amplitude $\boldsymbol{E}_o$ up to order $(k_0 d)^{-1}$ at $z = d$ in the form

$$\boldsymbol{E}_o(\boldsymbol{x},d) = \mathbb{O}^{(1)}\exp[ik_0 S_0]\,\boldsymbol{o}_0[\boldsymbol{k}'] + \mathbb{E}_0\theta_m a_y$$
$$\times \frac{n_e}{n_o}\frac{i\,\pi\{1 + \exp[i\,k_0\Phi(\boldsymbol{x})]\}}{k_0 d(n_e - n_o)}\sin(\boldsymbol{q}\cdot\boldsymbol{x})\boldsymbol{o}_0[\boldsymbol{k}'],$$
$$(52)$$

where $\Phi(\boldsymbol{x})$ has been given in Eq. (51). In analogy to Eq. (51) a twist in the director field ($a_y \neq 0$) leads again to a rotation of the incident polarization by $90°$.

The main result of the general discussion of amplitude solutions in Eqs. (44) is that one should observe, even for equilibrium patterns like the flexoelectric domains or the splay-twist Freedericksz pattern ($\boldsymbol{q} \parallel \hat{\boldsymbol{y}}$), diffraction fringes and shadowgraph pictures. This is confirmed in experiments [19,20,30,31]. By the way, restricting the light rays by apertures or using a slightly divergent light beam, as happens often in experiments, amounts, in principle, also to the generation of obliquely incident rays as well; this effect might be worth analyzing in the future. It is needless to say that an analysis based on pure geometric optics is unable to capture at all the optical properties of such patterns.

VII. CONCLUDING REMARKS

In the previous sections we have demonstrated that the optics of a nematic layer with a periodically distorted director field can be convincingly described by a standard short-wave length approximation. The new calculational scheme is much easier to handle than the previous treatments of the problem based on Fermat's principle and the summation of the phases along the ray path. It is not necessary to distinguish carefully between phase- and ray-refraction indices, which has, in fact, caused some problems in the earlier work. A big advantage

of our approach is that it covers from the beginning 3D experimental geometries where the polarization of the incident light and the orientation of the wave vector are arbitrary. Thus, for instance, the optical analysis of equilibrium patterns with $q \perp n_0$ is put on a firm basis. In general, it has been demonstrated that even a slight tilt of the nematic layer leads to a large increase of the intensity contrast of the patterns. Note that also a so called pretilt of the director at the confining substrates corresponds to an inclination of the cell. This feature is based on the phase-grating effects of patterns in the nematic layer and thus is not accessible in the framework of pure geometric optics. We have explicitly discussed only roll solutions characterized by a wave vector q. The generalization to patterns characterized by several wave vectors like squares or hexagons is straightforward. Furthermore, topological defects in the pattern, like dislocations, have their counterpart in the corresponding shadowgraph theory as well. It would be certainly a rewarding task to reconsider their present, highly sophisticated mathematical description. It is based on geometric optics and the ensuing caustics (see, e.g., Ref. [32] and references therein) and should be related to the present approach including diffraction effects.

In this work we have concentrated on the presentation of the theoretical method and, in particular, on analytical results to leading order in the distortion amplitude of the originally planar director configuration. This makes it possible to asses easily the impact of the various experimental input parameters. A detailed comparison with experimental results, in particular with those presented in Ref. [33], is planned in the near future.

APPENDIX A: REFLECTION AND REFRACTION IN HOMOGENEOUS NEMATIC LAYERS

The amplitude-grating effect of a distorted nematic layer depends on the amplitude $\mathbb{E}(x,d)$ ($\mathbb{O}(x,d)$) of the electric field component with extraordinary (ordinary) polarization at the upper surface, $z = d$ [see Eq. (28)]. The amplitudes depend linearly on their values $\mathbb{E}_0$ ($\mathbb{O}_0$) at $z = 0$. To make a full description of an experiment we have to connect the field amplitudes at the surfaces of the nematic layer to those of the adjacent glass plates.

According to Eq. (35) the wave vector $k = k_0 k' = k_0(k'_\parallel, k'_z)$ of the incident plane wave can be represented in terms of the polar angle ϑ_g and the azimuthal angle ϕ of k' with respect to the normal ($\parallel \hat{z}$) of the nematic layer. As indicated in Sec. II B we have to distinguish between TE waves with the polarization vector $p_E(k')$ and the TM ones with polarization vector $p_M(k')$ where

$$p_E(k') = \frac{\hat{z} \times k'}{|\hat{z} \times k'|}, \quad p_M(k') = \frac{p_E \times k'}{|p_E \times k'|}. \quad \text{(A1)}$$

In the case of perpendicular incidence ($\vartheta_g = 0$) we chose $p_E = \hat{y}$ and $p_M = \hat{x}$.

An incident TE wave with amplitude A and wave vector $k' = (k'_\parallel, k'_z > 0)$ is partially reflected [wave vector $k'_r =$

$(k'_\parallel, -k'_z)$] and partially transmitted at the glass-nematic interface. The parallel components of all participating wave vectors are the same, while $|k'_z|$ is determined by $|k'_\parallel|$ and the respective refraction indices n_{rf} in the different media as discussed in Sec. II B. The general procedure to calculate reflection and transmission of monochromatic plane waves at interfaces is discussed in many textbooks like [17]. A very clear presentation for nematics can be found in Ref. [34].

The continuity of the tangential electric field components leads at the glass-nematic interface $z = 0$ to the condition

$$A(t \cdot p_E(k')) = A\, t \cdot \{R_{EE}\, p_E(k') + R_{EM}\, p_M(k'_r) \\ + T_{Ee}\, e[k'_e] + T_{Eo}\, o[k'_o]\}, \quad \text{(A2)}$$

for both $t = \hat{x}$ and $t = \hat{y}$, where the nematic polarization vectors $e_0[k'_e], o_0[k'_o]$ are defined in Eq. (12). Note that according to Eq. (A2) an incident TE wave will, in general, produce TE and TM reflected waves with amplitudes given by the reflection coefficients R_{EE}, R_{EM}. In the nematic layer both ordinary and extraordinary contributions may exist with amplitudes determined by the transmission coefficients T_{Eo}, T_{Ee}. In analogy to Eq. (A2) we have to exploit in addition the continuity of the tangential magnetic field components on the basis of Eqs. (10). It turns out that the necessary continuity of the normal components of the magnetic induction $B = \mu H$ and of the dielectric displacement D is automatically fulfilled via the tangential conditions on E, H. Thus, we have to solve four linear equations for the coefficients R_{EE}, R_{EM} and T_{Ee}, T_{Eo}. Alternatively, if the incident field is parallel to p_M we need the reflection coefficients R_{MM}, R_{ME} and the transmission coefficients T_{Me}, T_{Mo}, which are defined in analogy to Eq. (A2). At the nematic-glass interface ($z = d$) we have to distinguish between incident ordinary plane waves ($E \parallel o_0$) and extraordinary ones ($E \parallel e_0$) [see Eq. (12)]. Thus, we have to calculate the corresponding reflection and transmission coefficients $R_{ee}, R_{eo}, T_{eE}, T_{eM}$ and, analogously, $R_{oe}, R_{oo}, T_{oE}, T_{oM}$. For instance, if the incident electric field is parallel to e_0, the continuity condition for the tangential electric field reads as follows:

$$A\,(t \cdot e_0[k'_e]) = A\, t \cdot \{R_{ee}\, e[(k'_e)_r] + R_{eo}\, o[(k'_o)_r] \\ + T_{eE}\, p_E(k') + T_{eM}\, p_M(k')\}. \quad \text{(A3)}$$

All reflection and transmission coefficients, which require only the solution of linear equations, depend on the azimuth and polar angles ϕ, ϑ_g of k' defined in Eq. (35) and on the refraction indices n_g, n_o, n_e. The resulting general final expressions are, however, lengthy and not very transparent; thus, they are not reproduced here. In the special case of perpendicular incidence ($\vartheta_g = 0$) we arrive at

$$\begin{aligned} (T_{Eo}, T_{Mo}) &= (\cos\phi, -\sin\phi)\frac{2n_g}{n_g + n_o}, \\ (T_{Me}, T_{Ee}) &= (\cos\phi, \sin\phi)\frac{2n_g}{n_g + n_e}, \\ (T_{oE}, T_{oM}) &= (\cos\phi, -\sin\phi)\frac{2n_o}{n_g + n_o}, \\ (T_{eM}, T_{eE}) &= (\cos\phi, \sin\phi)\frac{2n_e}{n_g + n_e}. \end{aligned} \quad \text{(A4)}$$

In experiments at most small ϑ_g are used. It then turns out that the transmission coefficients in Eqs. (A4) remain unchanged to order $O(\vartheta_g)$ and that only the wave vectors and the polarization vectors $\boldsymbol{p}_M$, $\boldsymbol{e}_0$ acquire corrections $\propto \sin(\vartheta_g)$. If we concentrate in addition on incident TM waves with $\phi = 0$, only the transmission coefficients T_{Me}, T_{eM} come into play. In other words, an incident TM wave leads only to a TM wave above the nematic layer.

In this Appendix we have concentrated on the electric field components of the electromagnetic field. Corresponding reflection and transmission coefficients can be defined as well for the magnetic field components. They are obtained, for instance, from the electric ones with the help of the Maxwell equations (4).

APPENDIX B: DETERMINATION OF THE FIELD AMPLITUDES

Here we sketch the derivation of the differential equations (see [18]) which determine the field amplitudes $\mathbb{O}$, $\mathbb{E}$, and $\mathbb{Z}$ in Eq. (28). First of all, the terms of the order $O(k_0^2)$ from $V = \boldsymbol{o}^N$ and $V = \boldsymbol{e}^N$ vanish by construction of the eikonal equations. Furthermore, $\boldsymbol{s}^N$ does not give a contribution in this order since $\boldsymbol{\nabla}(S_e + S_0)/2 \parallel \boldsymbol{s}^N$.

The terms $\propto k_0$ from Eq. (15) can be represented in terms of the vector operator $\boldsymbol{G}[\![V|S]\!] = (G_x, G_y, G_z)[\![V|S]\!]$ defined as

$$
\begin{aligned}
G_i[\![V|S]\!] &= 2(\partial_m S)(\partial_m V_i) + (\partial_m \partial_m S)V_i - \partial_i[(\partial_m S)V_m] \\
&\quad - (\partial_i S)(\partial_m V_m), \quad \text{with} \\
V &= (V_x, V_y, V_z) \text{ and } i, m = (x, y, z).
\end{aligned} \tag{B1}
$$

As usual, summation over the repeated indices m is assumed. Injecting the ansatz for $\boldsymbol{E}$ [Eq. (28)] into Eq. (7) and collecting all terms $\propto k_0$ we arrive at

$$
\begin{aligned}
&e^{i k_0 S_o} \boldsymbol{G}[\![\mathbb{O}\boldsymbol{o}^N|S_o]\!] + e^{i k_0 S_e} \boldsymbol{G}[\![\mathbb{E}\boldsymbol{e}^N|S_e]\!] \\
&\quad - e^{i k_0 (S_e + S_o)/2} \mathbb{Z}\, \boldsymbol{A} = 0, \quad \boldsymbol{A} := \boldsymbol{\varepsilon} \cdot \boldsymbol{s}^N.
\end{aligned} \tag{B2}
$$

In line with [18] the amplitude $\mathbb{Z}$ is now eliminated by multiplying the term with $i = x$ in Eq. (B2) with A_z and subtracting the term with $i = z$ after a multiplication with A_x. The analogous procedure is applied to the term with $i = y$. Thus, we arrive at the final equations to determine the field amplitudes $\mathbb{E}(\boldsymbol{x}, z)$, $\mathbb{O}(\boldsymbol{x}, z)$:

$$
\begin{aligned}
&A_z(a_- G_x[\![\mathbb{O}\boldsymbol{o}^N|S_o]\!] + a_+ G_x[\![\mathbb{E}\boldsymbol{e}^N|S_e]\!]) \\
&\quad = A_x(a_- G_z[\![\mathbb{O}\boldsymbol{o}^N|S_o]\!] + a_+ G_z[\![\mathbb{E}\boldsymbol{e}^N|S_e]\!]), \\
&A_z(a_- G_y[\![\mathbb{O}\boldsymbol{o}^N|S_o]\!] + a_+ G_y[\![\mathbb{E}\boldsymbol{e}^N|S_e]\!]) \\
&\quad = A_y(a_- G_z[\![\mathbb{O}\boldsymbol{o}^N|S_o]\!] + a_+ G_z[\![\mathbb{E}\boldsymbol{e}^N|S_e]\!]), \quad \text{where} \\
&a_\pm = \exp[\pm i k_0 (S_e - S_0)/2].
\end{aligned} \tag{B3}
$$

APPENDIX C: MAGNETIC FIELD IN ORDER $O(\theta_m^2)$

Expanding Eq. (31) to second order in θ_m^2 yields the following equation for the expansion coefficient $B^{(2)}(x, z)$ defined in Eq. (32):

$$
\begin{aligned}
&2n_e \frac{\partial}{\partial z} B^{(2)}(x, z) \\
&= -2\beta n_e \delta n_z(x, z) \left[B_0 \frac{\partial}{\partial z} \delta n_z(x, z) + \frac{\partial}{\partial x} B^{(1)}(x, z) \right] \\
&\quad - n_e \beta B^{(1)}(x, z) \frac{\partial}{\partial x} \delta n_z(x, z) \\
&\quad - B_0 \left[\frac{n_e^2}{n_o^2} \partial_{xx} S^{(2)}(x, z) + \partial_{zz} S^{(2)}(x, z) \right],
\end{aligned} \tag{C1}
$$

where $B^{(1)}(x, z)$ [Eq. (34)] and $S^{(2)}(x, z)$ [see Eq. (25)] have been already calculated. After straightforward integration over z we obtain

$$
B^{(2)}(x, d) = -B_0 \frac{q^2 d^2 \beta}{8\pi^2} \{2\beta + [6\beta + \pi^2(\beta + 1)]\cos(2qx)\}. \tag{C2}
$$

Furthermore, by collecting the terms $\propto \theta_m |\boldsymbol{k}'_\parallel|$ in Eq. (31) we obtain the following equation for $B_y^{(1,1)}$ in Eq. (32):

$$
\begin{aligned}
2n_e \frac{\partial}{\partial z} B^{(1,1)}(x, z) &= -B_0 n_e \beta \frac{\partial}{\partial x} \delta n_z - B_0 k'_x \beta \frac{\partial \delta n_z}{\partial z} \\
&\quad - 2\frac{k'_x n_e^2}{n_o^2} \frac{\partial B^{(1)}(x, z)}{\partial x} - B_0 \frac{n_e^2}{n_o^2} \frac{\partial^2 S^{(1,1)}}{\partial x^2} \\
&\quad - B_0 \frac{\partial^2 S^{(1,1)}}{\partial z^2}.
\end{aligned} \tag{C3}
$$

By performing the z integration within the one-mode approximation we arrive at

$$
B^{(1,1)}(x, d) = B_0 \beta \frac{n_e}{n_o^2 \pi} d^2 q^2 \cos(qx). \tag{C4}
$$

APPENDIX D: THE NEMATIC LAYER AS A DIFFRACTION GRATING

The electric field of the wave propagating through the nematic layer from $z = 0$ to $z = d$ is, in general, described by Eq. (28); finally, we need the field at the upper surface of the layer ($z = d$). At first we may safely neglect the $O(k_0^{-1})$ contribution $\propto \mathbb{Z}$. In any case, since $\boldsymbol{s}^N \approx \hat{\boldsymbol{z}}$ for small ϑ_g the associated plane waves propagate practically in the xy plane and are thus not relevant in the present context. With respect to the extraordinary and the ordinary field components their polarization vectors simplify considerably at $z = d$. We have $\boldsymbol{o}^N \to \boldsymbol{o}_0[\boldsymbol{k}'_o]$, $\boldsymbol{e}^N \to \boldsymbol{e}_0[\boldsymbol{k}'_e]$, where the constant vectors $\boldsymbol{o}_0, \boldsymbol{e}_0$ are defined in Eq. (12) with $\hat{\mathbf{n}} = \hat{\boldsymbol{x}}$. The corresponding amplitudes $\mathbb{O}, \mathbb{E}$, which are periodic in the horizontal coordinates $\boldsymbol{x} = (x, y)$, describe obviously the amplitude-grating effect of the nematic layer. In view of the general decomposition $S_e(\boldsymbol{x}, z) = S_e^0(\boldsymbol{x}, z) + \bar{S}_e(\boldsymbol{x}, z)$ [see Eqs. (19) and (23)], the phase-grating effect of the nematic layer is captured by $\bar{S}_e(\boldsymbol{x}, d)$. This quantity is periodic in $\boldsymbol{x}$ as well due to the sums of trigonometric functions with the arguments $\boldsymbol{q} \cdot \boldsymbol{x}$ and $2\boldsymbol{q} \cdot \boldsymbol{x}$ originating from the solutions of the eikonal equation Eq. (21) to order $O(\theta_m^2)$. For simplicity we concentrate on the discussion of the crucial extraordinary contribution $E_e(\boldsymbol{x})\boldsymbol{e}_0[\boldsymbol{k}'_e]$ with $E_e(\boldsymbol{x}) = \mathbb{E}(\boldsymbol{x}, d) \exp[i S_e(\boldsymbol{x}, d)]$.

The total electric field $\boldsymbol{E}_e(\boldsymbol{x},d)$ can be written as a Fourier series as

$$\boldsymbol{E}_e(\boldsymbol{x},d) = \mathbb{E}_0 \exp\left[i\,k_0\big(\boldsymbol{k}'_\parallel \cdot \boldsymbol{x} + k_z'^{,e}(\boldsymbol{k}')d\big)\right]$$
$$\times \sum_{n=-\infty}^{\infty} C^N(n) e^{i\,n(\boldsymbol{q}\cdot\boldsymbol{x})} \boldsymbol{e}_0[n\boldsymbol{q}/k_0 + \boldsymbol{k}'], \quad \text{(D1)}$$

where the phase prefactor comes from S_e^0 [Eq. (19)] evaluated at $z = d$. The Fourier series contains the product of the corresponding ones from $\mathbb{E}(\boldsymbol{x},d)$ and from the phase factor $\exp[i\,k_0 \bar{S}_e(\boldsymbol{x},d)]$. The latter can be transformed into a Fourier series by exploiting the identities

$$e^{i\,\alpha\sin\beta} = \sum_{n=-\infty}^{\infty} J_n(\alpha) e^{i\,n\beta}, \quad e^{i\,\alpha\cos\beta} = \sum_{n=-\infty}^{\infty} i^n J_n(\alpha) e^{i\,n\beta},$$
$$\text{(D2)}$$

where $J_n(\alpha)$ denotes the Bessel function of the first kind. For small θ_m, on which we mainly concentrate, it is sufficient to truncate the expansion of $\mathbb{E}(\boldsymbol{x},d)$ at θ_m, since, in general, $k_0\bar{S}_e(\boldsymbol{x},d)$ prevails. To obtain closed analytical expressions we use the one-mode approximation for $\delta\boldsymbol{n}$ [Eq. (3)] to evaluate the terms given in Eqs. (44a) and (26) and arrive at

$$\mathbb{E}(\boldsymbol{x},d) = \mathbb{E}_0[1 + c_{E1}\theta_m\sin(\boldsymbol{q}\cdot\boldsymbol{x})], \quad c_{E1} = (q_x d)\frac{\beta}{\pi},$$
$$k_0\bar{S}(\boldsymbol{x},d) = \cos(\boldsymbol{q}\cdot\boldsymbol{x})[\theta_m c_{S1} + \theta_m^2 c_{S2}\cos(\boldsymbol{q}\cdot\boldsymbol{x})], \quad \text{with}$$
$$c_{S1} = -(k_0 d)\frac{2\beta}{\pi}\cos(\phi)\sin(\vartheta_g), \quad c_{S2} = -(k_0 d)n_e\frac{\beta}{4}.$$
$$\text{(D3)}$$

Expanding thus the product $\mathbb{E}(\boldsymbol{x},d)\exp[i\,k_0\bar{S}_e(\boldsymbol{x},d)]$ we obtain the following approximations for the expansion coefficients $C^N(n)$ in Eq. (D1):

$$C^N(0) = 1 + \theta_m^2\frac{1}{4}\big[2i\,c_{S2} - c_{S1}^2\big],$$
$$C^N(\pm 1) = i\,\theta_m\frac{1}{2}(c_{S1} \mp c_{E1}), \quad \text{(D4)}$$
$$C^N(\pm 2) = i\,\theta_m^2\frac{1}{8}\big(2c_{S2} + ic_{S1}^2 \pm i\,2c_{S1}c_{E1}\big).$$

Since the electric field [Eq. (45)] at $z = d$ is expressed as a superposition of plane waves, it is easy to construct the electric field in the adjacent glass plate of thickness d_g. Here we have, in general, a superposition of TM and TE waves. Let us start with the TM waves which have the general representation

$$\boldsymbol{E}_M^G(\boldsymbol{x},z) = \mathbb{E}_0\exp[ik_0\boldsymbol{k}'_\parallel \cdot \boldsymbol{x}]\sum_{n=-\infty}^{\infty} C^G(n)\exp\big[i\big(n\boldsymbol{q}\cdot\boldsymbol{x}$$
$$+ zk_z^G(n)\big)\big]\boldsymbol{p}_M(\boldsymbol{k}' + n\boldsymbol{q}/k_0), \quad \text{for}$$
$$d \leqslant z \leqslant d + d_g \text{ with } k_z^G(n) = k_0\sqrt{n_g^2 - (\boldsymbol{k}'_\parallel + n\boldsymbol{q}/k_0)^2}.$$
$$\text{(D5)}$$

We are only interested in the propagating waves where the argument of the square root in Eq. (D5) is positive, which restricts the summation over n to a cutoff $n = n_{\text{cut}}$. For the typical nematic pattern $n_{\text{cut}} \gg 1$ holds. The contributions for $|n| > n_{\text{cut}}$ which decay exponentially with increasing z, i.e., the "evanescent" waves, are not recorded in the standard experiments. In fact, only the Fourier coefficients for small $|n| < 3$ will play an important role for small distortion amplitudes

θ_m. Thus, besides the leading terms in $|\boldsymbol{k}'_\parallel| = \sin(\vartheta_g)$ only the leading terms in the small quantity $|n\boldsymbol{q}|/k_0$ are kept in the following. For instance, the transmission coefficient T_{ga} from a glass to an air layer with refraction index $n_a = 1$ is given in this approximation as [17]

$$T_{ga}(\boldsymbol{k}' + n\boldsymbol{q}/k_0) = \frac{2n_g}{n_g + n_a}. \quad \text{(D6)}$$

Matching to the electric field [Eq. (45)] in the nematic layer at $z = d$ yields

$$C^G(n)\exp\left[i\,k_z^G(n)d\right] = T_{eM}(\boldsymbol{k}')\exp\left[i\,k_0 k_z'^{,e}(n)d\right]C^N(n). \quad \text{(D7)}$$

The wave gets then refracted again at the glass-air interface ($z = d + d_g$) and propagates further in air. The electric field in air for $z \geqslant d + d_g$ has the same representation as in Eq. (D5) with Fourier coefficients $C^A(n)$ instead of $C^G(n)$. Furthermore, we need $k_z^A(n)$, where n_g in $k_z^G(n)$ [Eq. (D5)] is replaced with $n_a = 1$. Matching the electric fields at $z = d + d_g$ yields

$$C^A(n)\exp\left[i\,k_z^A(n)(d + d_g)\right]$$
$$= T_{ga}(\boldsymbol{k}' + n\boldsymbol{q}/k_0)\exp\left[i\,k_0 k_z^G(n)(d + d_g)\right]C^G(n), \quad \text{(D8)}$$

where T_{ga} is given in Eq. (D6). At the end we arrive at

$$\boldsymbol{E}^A(\boldsymbol{x},z) = \mathbb{E}_0\exp[i\,k_0(\boldsymbol{k}'_\parallel \cdot \boldsymbol{x} + z')]\mathfrak{S}(\boldsymbol{x},z') \quad \text{with}$$
$$z' = z - (d + d_g) \quad \text{and} \quad \text{(D9)}$$

$$\mathfrak{S} = \sum_{n=-n_{\text{cut}}}^{n_{\text{cut}}} C_n^N T(d_g,n)\exp\big[i\,(n\boldsymbol{q}\cdot\boldsymbol{x}$$
$$+ i\,z'\big(k_z^A(n) - k_z^A(0)\big)\big]\boldsymbol{p}_M(\boldsymbol{k}' + n\boldsymbol{q}/k_0). \quad \text{(D10)}$$

The transfer function $T(d_g,n)$ describes the effect of the glass layer where

$$T(d_g,n) = T_{eM}(\boldsymbol{k}' + n\boldsymbol{q}/k_0)T_{ga}(\boldsymbol{k}' + n\boldsymbol{q}/k_0)$$
$$\times \exp\big[id_g k_z^G(n) + ik_z^e d\big]. \quad \text{(D11)}$$

The quantitative evaluation of the electric field in Eq. (D9) is straightforward but requires numerical effort. We concentrate here on analytical expressions which describe all relevant features for small θ_m very well. According to Eq. (D4) only the Fourier coefficients for $|n| < 3$ come into play. Exploiting in addition the smallness of ϑ_g and of $|n\boldsymbol{q}|/k_0$ all transmission coefficients like T_{ga} [see Eq. (D6)] can be safely replaced by their values for perpendicular incidence and can be taken out from the sums. An analogous approximation applies also to the polarization vectors. In the exponents of $\exp[i\,k_0(k_z^A(n) - k_z^A(0))z']$ we keep in the spirit of the paraxial approximation the leading term by expanding inside the square roots with respect to the small quantity q^2/k_0^2. Thus, starting from Eq. (D9) the electric field is well approximated by

$$\boldsymbol{E}^A(\boldsymbol{x},z) = \mathbb{E}_0 T_{eM} T_{ga}\exp[i\,k_0(\boldsymbol{k}'_\parallel \cdot \boldsymbol{x} + z')]\,\mathfrak{S}'(\boldsymbol{x},z')\,\boldsymbol{p}_M(\boldsymbol{k}'),$$
$$\text{(D12)}$$

$$\mathfrak{S}' = C^N(0) + \{C^N(+1)\exp[i\,\boldsymbol{q}\cdot\boldsymbol{x}]$$
$$+ C^N(-1)\exp[-i\,\boldsymbol{q}\cdot\boldsymbol{x}]\}\exp\left[-i\frac{q^2}{2k_0}z'\right]$$
$$+ \{C^N(+2)\exp[i\,2\boldsymbol{q}\cdot\boldsymbol{x}] + C^N(-2)\exp[-i\,2\boldsymbol{q}\cdot\boldsymbol{x}]\}$$
$$\times \exp\left[-i\frac{4q^2}{2k_0}z'\right] + \cdots, \tag{D13}$$

with $z' = z - (d + d_g)$. Hence, from the knowledge of the $C^N(n)$ we obtain immediately the relative intensity of the refraction fringes $I_n = |C^N(n)|^2$; the shadowgraph intensity $I_S(\boldsymbol{x},z)$ is determined by $|\boldsymbol{E}^A(\boldsymbol{x},z)|^2$. Further details are discussed in Sec. VI.

We have given explicit results for TM waves entering the nematic layer at $z = 0$ and leaving at $z = d$. In this case the coupling to the extraordinary waves in the nematic layer, which are associated with strong phase-grating effects, is guaranteed. This is the situation used in the typical experimental setups. The analysis of other cases, however, would follow step by step the same calculational scheme. We had to use the ordinary field amplitude $\mathbb{O}$ in the nematic layer which would then couple to the polarization vectors $\boldsymbol{p}_E$ in the glass plates.

APPENDIX E: DISCUSSION OF THE EIKONAL SOLUTIONS

In Sec. VI we demonstrated that an inclination of the wave vector $\boldsymbol{k}'$ of the incident light even by a small polar angle ϑ_g with respect to the $\hat{\boldsymbol{z}}$ [see Eq. (35)] may have a strong effect: Both the intensity of the first order ($n = 1$) diffraction fringes

and the contrast of the shadowgraph pictures considerably increase. So far we have concentrated on the terms linear in ϑ_g and to zero azimuthal angle ϕ. This section is devoted to the question of whether the use of a rotation of the incidence plane about the z axis, i.e., finite ϕ, will give additional advantages. For that purpose we have performed a systematic expansion of $\bar{S}_e$ [see Eq. (21)] up to second order in the director amplitude θ_m by using the following ansatz:

$$\bar{S}_e(\boldsymbol{x},z) = \theta_m S_e^{(1)}(\boldsymbol{x},z) + \theta_m^2 S_e^{(2)}(\boldsymbol{x},z). \tag{E1}$$

The expressions for the coefficients A, B [see Eq. (24)] to be used in Eq. (22) have not to be modified, but C has to be generalized as follows:

$$C = \theta_n\big(C_{\delta n}^1 + C_S^1\big) + \theta_n^2\big(C_{\delta n}^2 + C_S^2\big), \quad \text{with}$$
$$C_{\delta n}^1 = -2\beta k_z'^e\big(k_z'^e\delta n_z + k_y'\delta n_y\big),$$
$$C_S^1 = -\left[k_y'\frac{\partial}{\partial y}\bar{S}_e + (n_e/n_o)^2 k_x'\frac{\partial}{\partial x}\bar{S}_e\right],$$
$$C_{\delta n}^2 = \beta\big[-2k_y'k_z'^e\delta n_z\delta n_y + \big(k_x'^2 - k_y'^2\big)\delta n_y^2$$
$$+ (k_x')^2 - \big(k_z'^e\big)^2\delta n_z^2\big],$$
$$C_S^2 = -\left[2\beta k_x'\delta n_y + \frac{n_e^2}{n_o^2}\frac{\partial}{\partial y}\bar{S}_e\right]\frac{\partial}{\partial y}\bar{S}_e$$
$$-\left[\frac{\partial}{\partial x}\bar{S}_e + 2\beta(k_y'\delta n_y + k_z'^e\delta n_z)\right]\frac{\partial}{\partial x}\bar{S}_e, \tag{E2}$$

where $k_z'^e$ has been defined in Eq. (20).

To expand the eikonal solution of Eq. (22) to order $O(\theta_m^2)$ we need also the following relation:

$$\big(B + \sqrt{B^2 + AC}\big)^{-1} = \frac{1}{2k_z'^e}\left[1 - \theta_m\frac{4k_z'^e\beta k_x'\big(k_y'\delta n_y - k_z'^e\delta n_z\big) + k_y'\frac{\partial}{\partial y}\bar{S}_e - (n_e/n_o)^2\frac{\partial}{\partial x}\bar{S}_e}{4\big(k_z'^e\big)^2}\right]. \tag{E3}$$

In order θ_m we obviously arrive from Eq. (22) at the following differential equation for $S_e^{(1)}$:

$$\frac{\partial}{\partial z}S_e^{(1)} - \frac{1}{2k_z'^e}S_e^{(1)} - \left[k_y'\frac{\partial}{\partial y}S_e^{(1)} + (n_e/n_o)^2 k_x'\frac{\partial}{\partial x}S_e^{(1)}\right] = inh(\boldsymbol{x},z) \equiv C_{\delta n}^1. \tag{E4}$$

Let us now switch to Fourier space,

$$S_e^{(1)}(\boldsymbol{x},z) = \tilde{S}_e^{(1)}(\boldsymbol{q},z)\exp[i\,\boldsymbol{q}\cdot\boldsymbol{x}] + \text{c.c.}, \quad inh(\boldsymbol{x},z) = \widetilde{inh}(\boldsymbol{q},z)\exp[i\,\boldsymbol{q}\cdot\boldsymbol{x}] + \text{c.c.}, \tag{E5}$$

where $\partial_x \to i\boldsymbol{q}$ in Eq. (E4). Thus, the solution of Eq. (E4) in Fourier space with initial condition $\tilde{S}_e^1(\boldsymbol{q},z) = 0$ for $z = 0$ is easily obtained as

$$\tilde{S}_e^1(\boldsymbol{q},z) = \exp[-i\,\lambda(\boldsymbol{k}',\boldsymbol{q})z/d]\int_0^z dz'\exp[i\,\lambda(\boldsymbol{k}',\boldsymbol{q})(z'/d)]\,\widetilde{inh}(\boldsymbol{q},z'), \quad \text{with} \quad \lambda(\boldsymbol{k}',\boldsymbol{q}) = -\frac{k_y'q_y + (n_e/n_o)^2 k_x'q_x}{k_z'^e}. \tag{E6}$$

Returning to position space we arrive from Eq. (E5) at the following general representation of $S_e^{(1)}(\boldsymbol{x},z)$:

$$S_e^{(1)}(\boldsymbol{x},z) = \sin(\boldsymbol{q}\cdot\boldsymbol{x})f_s(z) + \cos(\boldsymbol{q}\cdot\boldsymbol{x})f_c(z). \tag{E7}$$

The functions f_s, f_c are obtained by performing the z integration in Eq. (E6). Their specific form depends on the ansatz chosen for δn_y, δn_z, which, according to Eq. (E4), determines $inh(\boldsymbol{x},z)$. The phase-grating effect to order θ_m is determined by $S_e^{(1)}(\boldsymbol{x},z = d)$; an explicit analytical expression is obtained again within the one-mode approximation for δn [Eq. (3)]. The lengthy expressions simplify considerably if we confine ourselves in addition to the leading terms in $|\boldsymbol{k}'_\parallel| \propto \sin(\vartheta_g)$ according to Eq. (35), which give already the main insight into the relevance of the various contributions to $S_e^{(1)}(\boldsymbol{x},z = d)$. Restricting ourselves to the terms up to order $O(\sin^2(\vartheta_g))$ leads to

$$f_s(d) = -d\beta\frac{\sin(\vartheta_g)^2\cos\phi}{n_o^2\pi}\big(2n_o^2 a_y\sin\phi + n_e^2\cos\phi q_x + n_o^2\sin\phi q_y\big) + O(\sin^4(\vartheta_g)), \tag{E8}$$

$$f_c(d) = -d\frac{2\beta}{\pi}\sin(\vartheta_g)\cos\phi + O(\sin^3(\vartheta_g)). \tag{E9}$$

Note that in the case of perpendicular incidence ($k'_x, k'_y = 0$) we get no contribution to phase grating of the order $O(\theta_m)$; the only linear contribution $\propto k'_x \cos(\boldsymbol{q}\cdot\boldsymbol{x})$, which we have obtained already before in Eq. (25), requires a nonzero ϑ_g. We have to keep the terms $\propto\sin^2(\vartheta_g)$ in order to identify a contribution of the director twist($\sim\delta n_y$) and of an in-plane rotation of $\boldsymbol{q}$ (finite q_y).

Turning to quadratic order in θ_m^2 we have to solve an equation for the expansion coefficient $\bar{S}_e^{(2)}$ [see Eq. (E1)], which has the same structure as Eq. (E4) except a different inhomogeneity $inh_2(\boldsymbol{x},z)$. Here we have contributions from $C_{\delta n}^2$ and from C_S^2 in Eq. (E2) where the solution $\bar{S}_e^{(1)}$ given in Eq. (E7) has to be used. In addition, we find contributions from the product of the term $\propto\theta_m$ in Eq. (E3) and the terms $(C_{\delta n}^1 + C_S^1)$ in Eq. (E2). In analogy to the treatment of Eq. (E4) and its solution shown in Eq. (E8) we arrive at the following general representation for $S_e^{(2)}(\boldsymbol{x},z)$:

$$S_e^{(2)}(\boldsymbol{x},z) = f_0^{(2)}(z) + \sin(2\boldsymbol{q}\cdot\boldsymbol{x})f_s^{(2)}(z) + \cos(2\boldsymbol{q}\cdot\boldsymbol{x})f_c^2(z). \tag{E10}$$

Performing the required z integrations within the one-mode approximation and restricting ourselves to the terms up to order $O(\sin(\vartheta_g))$ the analytical expressions read as follows:

$$f_0^{(2)}(d) + f_c^{(2)}(d)\cos(2\boldsymbol{q}\cdot\boldsymbol{x}) = -d\frac{n_e\beta}{8}[1 + \cos(2\boldsymbol{q}\cdot\boldsymbol{x})] + O(\sin^2(\vartheta_g)), \tag{E11a}$$

$$f_s^{(2)}(d) = -d\beta\frac{\sin(\vartheta_g)}{8}\left[a_y\sin\phi + 4\left(\beta/\pi^2 + \frac{n_e^2}{n_o^2}\right)q_x d\cos\phi + q_y d\sin\phi\right] + O(\sin^3(\vartheta_g)). \tag{E11b}$$

As to be expected the ϑ_g-independent contribution to $S_e^{(2)}(\boldsymbol{x},d)$ in Eq. (E11a) is equal to the one already derived in Eq. (26). The new term $f_s^{(2)}(d) \propto \sin(\vartheta_g)$ reveals the impact of a director twist ($\sim\delta n_y$) and of an in-plane rotation of $\boldsymbol{q}$ (finite q_y).

APPENDIX F: SHADOWGRAPHY IN OPTICALLY ISOTROPIC MEDIA: RAYLEIGH-BÉNARD CONVECTION

In the following we comment briefly on the short-wavelength expansion technique for RBC, where we follow closely the notations in Ref. [10]. The convection cell has the thickness d ($0 < z < d$) with $T_1 > T_2$ the prescribed temperatures at the lower and upper plates, respectively. In the convective state the temperature distribution is given as

$$T(x,y,z) = T_0 - \Delta T\frac{z - d/2}{d} + \Theta_{conv}(x,y,z), \quad\text{with}$$
$$T_0 = \frac{T_1 + T_2}{2}, \Delta T = (T_1 - T_2), \tag{F1}$$

where $\Theta_{conv}(x,y,z)$ denotes the convective temperature contribution, which is available as a Galerkin expansion from standard codes. We consider the fluid as an isotropic medium with a space-dependent dielectric permeability $\varepsilon(x,y,z)$ and constant magnetic permeability $\mu = 1$. Thus, the refraction index is given as $n^2 = \varepsilon$. It depends on the density ρ, which varies with temperature in the RBC case. Thus, we use an expansion about the mean temperature T_0:

$$n(\rho(T)) = n(\rho(T_0)) + \frac{\partial n}{\partial\rho}\frac{d\rho}{dT}\bigg|_{T=T_0}$$
$$\times\left[-\Delta T\frac{z - d/2}{d} + \Theta_{conv}(x,y,z)\right]. \tag{F2}$$

According to Eq. (1) in Ref. [10], the three terms on the right-hand side of Eq. (F2) are parametrized as

$$n(x,y,z) = n_0 + n_{heat} + n_{conv}$$
$$\equiv n_0 + n_2 z/d + n_1\sum_{i=1}^{N} a_i(x,y)b_i(z), \tag{F3}$$

where the coefficients n_0, n_2 describe the heat conduction state, while the term $n_{conv} \propto n_1$ measures the overall amplitude of Θ_{conv} in an Galerkin expansion.

The starting point for the optical analysis of RBC patterns are the following wave equations for the electric field, $\boldsymbol{E}$, and the magnetic field, $\boldsymbol{H}$, with a monochromatic time dependence (see, e.g., Eqs. (5) and (6) in Ref. [17]):

$$\Delta\boldsymbol{E} + n^2 k_0^2\boldsymbol{E} + 2\nabla[\boldsymbol{E}\cdot\nabla(\ln n)] = 0,$$
$$\Delta\boldsymbol{H} + n^2 k_0^2\boldsymbol{H} + 2[\nabla(\ln n)]\wedge\text{rot}\boldsymbol{H} = 0. \tag{F4}$$

The short-wave expansion is based on the ansatz:

$$\boldsymbol{E} = \boldsymbol{e}(\boldsymbol{r})\exp[ik_0 S(\boldsymbol{r})],$$
$$\boldsymbol{H} = \boldsymbol{h}(\boldsymbol{r})\exp[ik_0 S(\boldsymbol{r})], \tag{F5}$$
$$\boldsymbol{r} = (x,y,z).$$

It is easy to see that we arrive from Eqs. (F4) at the following equations for $\boldsymbol{e}(\boldsymbol{r})$ [see [17], Eqs. (16)]:

$$[n^2 - (\nabla S)^2]\boldsymbol{e} - \frac{1}{ik_0}\boldsymbol{L}^e(\boldsymbol{e},S,n) = 0 + O(k_0^{-2})\quad\text{with}$$
$$\boldsymbol{L}^e(\boldsymbol{e},S,n) = \{\Delta S\,\boldsymbol{e} + 2[\boldsymbol{e}\cdot\nabla(\ln n)]\}\nabla S + 2(\nabla S\cdot\nabla)\cdot\boldsymbol{e}. \tag{F6}$$

Analogously we obtain

$$[n^2 - (\nabla S)^2]\boldsymbol{h} - \frac{1}{ik_0}\boldsymbol{L}^h(\boldsymbol{e},S,n) = 0 + O(k_0^{-2}),\quad\text{with}$$
$$\boldsymbol{L}^h(\boldsymbol{h},S,n) = [2\nabla S\cdot\nabla(\ln n) - \Delta S]\boldsymbol{h}$$
$$- 2[\boldsymbol{h}\cdot\nabla(\ln n)]\nabla S - 2[\nabla S\cdot\nabla]\boldsymbol{h}. \tag{F7}$$

The leading order terms in Eqs. (F6) and (F7) yield the eikonal equation

$$(\nabla S(x,y,z))^2 = n^2(x,y,z), \tag{F8}$$

while the next order terms $(\propto k_0^{-1})$ determine the amplitudes $\boldsymbol{e}, \boldsymbol{h}$, respectively [see the remarks in Ref. [17] before Eqs. (41) and (42) there].

We here only address a 2D configuration with perpendicularly incident light in analogy to the planar case in nematics. The refraction index [Eq. (F3)] varies thus only in the xz plane ("convection rolls"), which is also the incidence plane of the light with $\boldsymbol{H}$ in the y direction. In the heat conduction state [$a_i \equiv 0$ in Eq. (F3)] we obtain immediately

$$S \equiv S^{(0)}(z) = n_0 z + n_2 \frac{z^2}{2d}. \qquad (F9)$$

Using the ansatz,

$$S = S^0 + S' \equiv S^0 + n_1 S^{(1)} + n_1^2 S^{(2)} + \cdots \qquad (F10)$$

and Eq. (F3) for the refraction index, we obtain from Eq. (F8)

$$2(n_0 + n_2 z)\frac{\partial}{\partial z} S'(x,z) + \left(\frac{\partial}{\partial x} S'\right)^2$$
$$= 2(n_0 + n_2 z)n_{\text{conv}}(x,z) + n_{\text{conv}}^2(x,z). \qquad (F11)$$

In linear order in n_1, n_2 Eq. (F11) can be directly solved and we arrive at

$$S^{(1)}(x,z) = \sum_{i=1}^{N} \frac{\partial}{\partial x} a_i(x) \int_0^z dz' b_i(z'). \qquad (F12)$$

Thus, in contrast to the nematic case we obtain already phase modulation in first order in n_1. For the solution $S^{(2)}$ proportional to n_1^2 [see Eq. (F10)] we obtain from Eq. (F11) the following ODE in z:

$$n_1^2 \left\{ 2(n_0 + n_2 z)\frac{\partial}{\partial z} S^{(2)}(x,z) + \left[\frac{\partial}{\partial x} S^{(1)}(x,z)\right]^2 \right.$$
$$\left. + \left[\frac{\partial}{\partial z} S^{(1)}(x,z)\right]^2 \right\} = n_{\text{conv}}^2(x,z), \qquad (F13)$$

which can be solved by a simple z integration. Note that the term n_{conv}^2 on the right-hand side of Eq. (F13) cancels against $[\partial_z S^{(1)}]^2$. Thus, we arrive at

$$S^{(2)}(x,d) = -\frac{1}{2n_0} \int_0^d dz \sum_{i,j} \frac{da_i(x)}{dx} \frac{da_j(x)}{dx}$$
$$\times \int_0^z \int_0^z dz' dz'' b_i(z') b_j(z'). \qquad (F14)$$

Consequently, one obtains the following expression for the total phase modulation term at $z = d$:

$$k_0 S(x,y,d) = k_0 S^{(0)} + k_0 n_1 S^{(1)}(x,d) + k_0 n_1^2 S^{(2)}(x,d). \qquad (F15)$$

When we use the expression for $S^{(0)}, S^{(1)}, S^{(2)}$ given in Eqs. (F9), (F12), and (F14) we agree with Ref. [10] up to order n_1 but disagree in order n_1^2, though the corresponding terms look very similar.

Since $\boldsymbol{h}$ has only a nonzero y component, h_y, in the present geometry, it is convenient to determine the amplitude modulation contribution from Eq. (F7). To order k_0^{-1}, starting from Eq. (F7), we have to solve $\boldsymbol{L}^h h_y(x,z) = 0$ with the boundary condition $h_y(x,0) = H_0$, where H_0 denotes the magnetic field amplitude of the incident plane wave. Explicitly written, $\boldsymbol{L}^h h_y = 0$ reads as follows:

$$\left\{ 2\left[\frac{\partial}{\partial x} S \frac{\partial}{\partial x} + \frac{\partial}{\partial z} S \frac{\partial}{\partial z}\right] \ln(n(x,z)) - \Delta S \right\} h_y(x,z)$$
$$- 2\left[\frac{\partial}{\partial x} S \frac{\partial}{\partial x} + \frac{\partial}{\partial z} S \frac{\partial}{\partial z}\right] h_y(x,z) = 0. \qquad (F16)$$

Note that the term $\nabla S[\boldsymbol{h} \cdot \nabla \ln n]$ in Eq. (F7) has vanished identically. Equation (F16) is solved iteratively by using the ansatz:

$$h_y(x,z) = h_y^0(z) + n_1 h_y^{(1)}(x,z) + O(n_1^2). \qquad (F17)$$

In the heat conducting state (no x dependence) Eq. (F16) simplifies to

$$\left[\frac{\partial}{\partial z} S \frac{\partial}{\partial z} \ln(n_0 z + n_2 z^2/d) - \partial_{zz} S\right] h_y(z)$$
$$- 2\frac{\partial}{\partial z} S \frac{\partial}{\partial z} h_y(x,z) = 0, \qquad (F18)$$

where only $S \equiv S_0$ from Eq. (F9) has to be used. Neglecting the small n_2 contributions (they can easily be incorporated), the incident amplitude is not modified, i.e., $h_y^0(z) \equiv H_0$. In order n_1 we obtain the ODE,

$$2n_0 \frac{\partial}{\partial z} h^{(1)}(x,z) = -\partial_{xx} S^{(1)}(x,z), \qquad (F19)$$

which leads to the following magnetic field amplitude:

$$h_y(x,z) = H_0 \left\{ 1 - \frac{n_1}{2n_0} \sum_{i=1}^{N} \left[\frac{\partial^2}{\partial x^2} a_i(x)\right] \int_0^z dz' b_i(z') \right\}. \qquad (F20)$$

This expression agrees perfectly with Eq. (20) in Ref. [10]. Thus, it has been proven that also in RBC our calculational scheme needs only a few systematic steps to reproduce the previous results in [10], which were obtained after tedious calculations. It is obvious that oblique incidence can be treated without difficulty within our calculational scheme as well. Also proceeding to higher order terms in n_1 is straightforward.

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