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Rotating magnetic field effect on an onset of convection in a horizontal layer of conducting fluid

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HIGHLIGHTS

- Specific instability mechanism related to the magnetic field inhomogeneity is found.
- The instability threshold depends non-monotonically on the inhomogeneity parameter.
- This mechanism can lead to instability even for the potentially stable stratification.

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ABSTRACT

The onset of convection in a horizontal layer of an electrically conducting fluid heated from below is studied in the presence of a horizontal magnetic field rotating about a vertical axis. Two different ranges of the rotation frequency are considered: (1) the case of high frequencies of rotation for which the averaging approach is used and (2) the case of finite frequencies for which the averaging approach might be implemented. It is shown that in general the magnetic field stabilizes the static state of pure thermal conduction. However, there exists a parameter range where the inhomogeneity of the magnetic field is strong enough to provide energy for the growth of disturbances even if the fluid layer is stably stratified.

1. Introduction

The problem of the onset of convection in a plane horizontal fluid layer heated from below is a classical problem of hydrodynamic stability theory. There are also many works on the effect of various external fields on the onset of convection. The present paper deals with the investigation of the onset of convection in infinite plane horizontal layer of electrically conducting fluid subjected to a rotating magnetic field. It is known that in the case of a finite size cavity a rotating magnetic field induces an azimuthal flow in the fluid. In infinite layers, for fluids of not too high electrical conductivity, rotation is absent.

In early works on the effect of magnetic fields on the onset of convection in electrically conducting fluids by Thompson [1] and Chandrasekhar [2–4], plane horizontal layers subjected to the uniform static magnetic fields were considered. It was shown that the effects of vertical and horizontal components of magnetic field are different. The boundary of monotonous instability is affected only by the vertical component of the magnetic field. It leads to the growth of the critical Rayleigh number and the critical wave number, i.e. magnetic field increases the stability of the conductive state and shifts the instability to shorter wavelength perturbations. Moreover, the property of isotropy and degeneracy (the possibility of the existence of convective patterns both in the form of rolls and in the form of the superposition of the several rolls) related to that isotropy, are kept in the case of a purely vertical magnetic field. Contrary to that, the presence of a horizontal component of the magnetic field breaks the symmetry and convection arises in the form of rolls with axes parallel to them. But the stability threshold remains unchanged. In other words, convective rolls which are not parallel to magnetic field are damped. In the papers mentioned above it was found that vertical magnetic field may lead to an oscillatory instability. It is known that in the absence of magnetic field

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oscillatory perturbations are damped. Chandrasekhar found that the necessary condition for oscillatory instability is the inequality $\chi > \nu_m$, where ν_m is magnetic diffusivity and χ is thermal diffusivity. Moreover, the oscillatory instability occurs only for sufficiently strong fields. Nakagawa [5,6] verified experimentally some of these theoretical predictions. He confirmed that a vertical magnetic field increases the stability of a conductive state and that with increasing magnetic field the horizontal wavelength of convective cells decreases. The conclusion that horizontal magnetic field does not influence the threshold of instability was confirmed in the experiments of Lehnert and Litlle [7]. In this paper it was shown that in the horizontal magnetic field convection arises in the form of rolls with axes parallel to the magnetic field.

Gershuni and Zhukhovitsky [8] considered the problem of the stability of the conductive state of an electrically conducting fluid in a plane vertical layer with a fixed constant vertical temperature gradient imposed on the boundaries, under the influence of horizontal magnetic field. A general theory of the perturbation spectra and the conductive state instability under magnetic fields was developed in the works by Sorokin & Sushkin [9] and Shliomis [10,11].

In [12] supercritical convection in a horizontal layer in the presence of a vertical magnetic field is studied. The influence of twisted magnetic field on thermal convection was studied in the paper [13] by Busse and Pesch.

A rotating magnetic field was widely studied as the source of MHD-flows in cylinders of finite and infinite heights. Gelfgalt and Priede presented a detailed review on this topic [14]. The influence of a rotating magnetic field on the convection was considered in [15–25]. In [15–17,21,22,25,26] the effect of rotating magnetic field on crystal growth was investigated. The possibility to obtain the crystals with improved properties applying RMF was analyzed. In [18] the effect of quickly rotating homogeneous magnetic field on the onset of convection in the horizontal fluid layer was investigated. In this case it was possible to neglect the angular motion of fluid excited by magnetic field. In the present paper we continue investigations of the article [18]. We investigate the case of inhomogeneous magnetic field.

The paper is organized as follows. In the Section 2 basic equations and boundary conditions are discussed. The Section 3 concerns the limit case of small dissipative hydrodynamic and thermal time scales, when it is possible to prove and develop the averaging method. The case of finite field rotation frequencies is considered in Section 4.

2. Governing equations

We consider a convective flow in the infinite horizontal layer of an electrically conducting fluid heated from below and subjected to a magnetic field. It is assumed that the exterior of the layer is insulated and the magnetic field applied far away from the layer is parallel to it and rotates uniformly about a vertical axis. Inside the layer and in its vicinity the magnetic field is inhomogeneous, but it depends on the vertical coordinate only (in the absence of the flow). As shown below, such an inhomogeneity is caused by the finite rate of diffusion in the conducting layer. The goal of the present paper is to study the effect of magnetic fields on the onset of convection.

Let us introduce a Cartesian coordinate system x, y, z in such a way that the z-axis is opposite to the direction of gravity, and the origin of the coordinate system is located on the midplane of the layer. The governing equations based on the magnetohydrodynamic approach and the Boussinesq approximation are:

$$\frac{\partial \vec{B}}{\partial t} = \nu_m \nabla^2 \vec{B} + \nabla \times (\vec{v} \times \vec{B}), \tag{1}$$

$$\nabla \cdot \vec{B} = 0, \tag{2}$$

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \vec{v} + g\beta T \vec{e}_z + \frac{1}{\mu\rho}\vec{B}\cdot\nabla\vec{B}, \tag{3}$$

$$\frac{\partial T}{\partial t} + \vec{v}\cdot\nabla T = \chi \nabla^2 T, \tag{4}$$

$$\nabla \cdot \vec{v} = 0, \tag{5}$$

where $\nu_m = 1/(\sigma\mu)$ is the magnetic diffusivity. The other notations are conventional.

It is assumed that the layer boundaries are characterized by perfect thermal conductivity and vanishing electrical conductivity, and the magnetic permeability μ is constant. We consider the case of rigid boundaries, as well as the case of tangential stress-free boundaries.

In the first case the boundary conditions are:

$$z = 0, \quad L : \vec{v} = 0, \quad T = \theta,\ 0, \quad (\nabla \times \vec{B}) \cdot \vec{e}_z = 0, \quad \vec{B} = \vec{B}^{(o)}, \quad |z| \to \infty : \quad B_x^{(o)} = B_* \cos\omega t, \quad B_y^{(o)} = B_* \sin \omega t, \tag{6}$$

and in the second case

$$z = 0, \quad L : v_z = 0, \quad \frac{\partial v_{x,y}}{\partial z} = 0, \quad T = \theta,\ 0, \quad (\nabla \times \vec{B}) \cdot \vec{e}_z = 0, \quad \vec{B} = \vec{B}^{(o)}, \quad |z| \to \infty : \quad B_x^{(o)} = B_* \cos\omega t, \quad B_y^{(o)} = B_* \sin\omega t. \tag{7}$$

Here, $\vec{B}^{(o)}$ is the magnetic field induction outside the layer, and θ is the temperature differences on the boundaries of layer. The magnetic permeability outside the layer is assumed to be equal to that inside the layer.

We choose the following scales for length, time, magnetic field induction, velocity, temperature and pressure:

$$L, \ \frac{L^2}{\chi}, \ \frac{\chi B_*}{\nu_m}, \ \frac{\chi}{L}, \ \theta, \ \frac{\rho\nu\chi}{L^2},$$

where L is the layer thickness.

The equations in the dimensionless form read:

$$\frac{\mathrm{Pr}_m}{\mathrm{Pr}}\frac{\partial \vec{B}}{\partial t} = \nabla^2 \vec{B} + \frac{\mathrm{Pr}_m}{\mathrm{Pr}}\nabla \times (\vec{v} \times \vec{B}), \tag{8}$$

$$\nabla \cdot \vec{B} = 0, \tag{9}$$

$$\frac{1}{\mathrm{Pr}}\left(\frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v}\right) = -\nabla p + \nabla^2 \vec{v} + \mathrm{Ra}\, T\vec{e}_z + \frac{\mathrm{Ha}^2\,\mathrm{Pr}_m}{\mathrm{Pr}}\vec{B}\cdot\nabla\vec{B}, \tag{10}$$

$$\frac{\partial T}{\partial t} + \vec{v}\cdot\nabla T = \nabla^2 T, \tag{11}$$

$$\nabla \cdot \vec{v} = 0. \tag{12}$$

The boundary conditions in the dimensionless form in the first case are

$$z = 0, 1 : \quad \vec{v} = 0, \quad T = 1,\ 0, \quad (\nabla \times \vec{B}) \cdot \vec{e}_z = 0, \quad \vec{B} = \vec{B}^{(o)}, \quad |z| \to \infty : \quad B_x^{(o)} = \frac{\mathrm{Pr}}{\mathrm{Pr}_m}\cos\omega t, \quad B_y^{(o)} = \frac{\mathrm{Pr}}{\mathrm{Pr}_m}\sin\omega t, \tag{13}$$

and in the second case

$$z = 0, 1 : \quad v_z = 0, \quad \frac{\partial v_{x,y}}{\partial z} = 0, \quad T = 1,\ 0, \quad (\nabla \times \vec{B}) \cdot \vec{e}_z = 0, \quad \vec{B} = \vec{B}^{(o)}, \quad |z| \to \infty : \quad B_x^{(o)} = \frac{\mathrm{Pr}}{\mathrm{Pr}_m}\cos\omega t, \quad B_y^{(o)} = \frac{\mathrm{Pr}}{\mathrm{Pr}_m}\sin\omega t. \tag{14}$$

The problem is characterized by the following dimensionless parameters:

$$\mathrm{Ha} = \frac{B_* L}{\sqrt{\nu\rho\mu\nu_m}}, \qquad \mathrm{Ra} = \frac{g\beta\theta L^3}{\nu\chi}, \qquad \mathrm{Pr} = \frac{\nu}{\chi},$$
$$\Omega = \frac{\omega L^2}{\chi}, \qquad \mathrm{Pr}_m = \frac{\nu}{\nu_m},$$

where Ha is the Hartmann number, Ra is the Rayleigh number, Pr is the Prandtl number, Ω is the dimensionless frequency of magnetic field rotation, and Pr_m is the magnetic Prandtl number.

3. Base state

Eqs. (8)–(12) lead to the solution satisfying the motionless state conditions:

$$\vec{v}_0 = 0, \qquad T_0 = 1 - z, \tag{15}$$
$$\vec{B}_0 = \frac{\mathrm{Pr}}{\mathrm{Pr}_m}\mathrm{Re}(\vec{b}_0 e^{i\Omega t}), \qquad \vec{B}_0^{(o)} = \frac{\mathrm{Pr}}{\mathrm{Pr}_m}\mathrm{Re}(\vec{b}_0^{(o)} e^{i\Omega t}),$$
$$\vec{b}_0 = (\vec{e}_x - i\vec{e}_y)\frac{\cosh\left(\gamma(1+i)\left(z-\frac{1}{2}\right)\right)}{\cosh\left(\frac{\gamma(1+i)}{2}\right)}, \quad \vec{b}^{(o)} = \vec{e}_x - i\vec{e}_y, \tag{16}$$

where a new dimensionless parameter $\gamma = \sqrt{\frac{\mathrm{Pr}_m\,\Omega}{2\,\mathrm{Pr}}}$ is introduced. This parameter defines the depth of magnetic field penetration into the fluid (the penetration depth is $1/\gamma$).

It follows from (16) that in the base state the magnetic field inside the layer is horizontal and uniform in horizontal directions. It rotates with the same angular velocity as the imposed magnetic field. Moreover, the magnetic field magnitude and the angle between the magnetic field at some point in the layer and the imposed magnetic field depend on the vertical coordinate, but do not depend on time. Outside the layer the magnetic field is homogeneous: since the distribution of induced currents turns out to be antisymmetric with respect to the midplane of the layer, the fields generated by them outside the layer are mutually compensated. In the limit case of vanishing γ the magnetic field inside the layer becomes homogeneous and coincides with the magnetic field outside the layer. In the opposite limit case of large γ the magnetic field inside the layer is concentrated in thin skin-layers.

4. Linear stability of base state

In this section we analyze the stability of the base state (15)–(16) with respect to small perturbations. Using the same notations for perturbations as for the full variables $\vec{v}$, p, T, $\vec{B}$ and linearizing Eqs. (8)–(12) about the base state (15)–(16), we obtain the equations for small perturbations in the form:

$$\frac{\mathrm{Pr}_m}{\mathrm{Pr}}\frac{\partial \vec{B}}{\partial t} = \nabla^2\vec{B} + \frac{\mathrm{Pr}_m}{\mathrm{Pr}}\nabla\times(\vec{v}\times\vec{B}_0), \tag{17}$$
$$\nabla\cdot\vec{B} = 0, \tag{18}$$
$$\frac{1}{\mathrm{Pr}}\frac{\partial\vec{v}}{\partial t} = -\nabla p + \nabla^2\vec{v} + \mathrm{Ra}\,T\vec{e}_z + \frac{\mathrm{Ha}^2\,\mathrm{Pr}_m}{\mathrm{Pr}}(\vec{B}_0\cdot\nabla\vec{B} + \vec{B}\cdot\nabla\vec{B}_0), \tag{19}$$
$$\frac{\partial T}{\partial t} = \nabla^2 T + \mathrm{v}_z, \tag{20}$$
$$\nabla\cdot\vec{v} = 0. \tag{21}$$

4.1. The case of high frequencies of magnetic field rotation

Consider first the case of high frequencies of magnetic field rotation when the period of magnetic field rotation is small in comparison with the dissipative viscous and thermal time scales $\Omega \gg \mathrm{Pr}$, $\Omega \gg 1$. It is convenient here to apply an averaging procedure. For this purpose we represent the perturbations of the magnetic field in the form $\vec{B} = \mathrm{Re}(\vec{b}e^{i\Omega t})$. This yields the following equations:

$$2i\gamma^2\vec{b} = \nabla^2\vec{b} + \nabla\times(\vec{v}\times\vec{b}_0), \tag{22}$$
$$\nabla\cdot\vec{b} = 0, \tag{23}$$
$$\frac{1}{\mathrm{Pr}}\frac{\partial\vec{v}}{\partial t} = -\nabla p + \nabla^2\vec{v} + \mathrm{Ra}\,T\vec{e}_z + \frac{1}{2}\mathrm{Ha}^2 Re(\vec{b}_0^*\cdot\nabla\vec{b} + \vec{b}\cdot\nabla\vec{b}_0^*), \tag{24}$$
$$\frac{\partial T}{\partial t} = \nabla^2 T + v_z, \tag{25}$$
$$\nabla\cdot\vec{v} = 0. \tag{26}$$

Write down the equation for the vertical component of the magnetic field as:

$$2i\gamma^2 b_z = \nabla^2 b_z + \Lambda \mathrm{v}_z, \tag{27}$$

where $\Lambda = \vec{b}_0\cdot\nabla$.

Applying double curl operation to the Eq. (24) and projecting it on vertical direction, we have:

$$\frac{1}{\mathrm{Pr}}\frac{\partial\nabla^2\mathrm{v}_z}{\partial t} = \nabla^4\mathrm{v}_z + \mathrm{Ra}\,\Delta_2 T - \frac{1}{2}\mathrm{Re}\left(\mathrm{Ha}^2\left(\frac{d^2\Lambda^*}{dz^2} - \Lambda^*\nabla^2\right)b_z\right). \tag{28}$$

Taking into account the circular polarization of the magnetic field ($b_{0x} = -ib_{0y}$), we write down the operator Λ in the form

$$\Lambda = FN, \tag{29}$$

where N is the differential operator of circular polarization

$$N = \frac{\partial}{\partial x} - i\frac{\partial}{\partial y}, \tag{30}$$

having the property

$$NN^* = \Delta_2. \tag{31}$$
$$F = \frac{\cosh(\gamma(1+i)(z-1/2))}{\cosh(\gamma(1+i)/2)}. \tag{32}$$

Eq. (27) takes the form

$$2i\gamma^2 b_z = \nabla^2 b_z + FN\mathrm{v}_z. \tag{33}$$

It is convenient to represent the solution to this equation in the form

$$b_z = NS, \tag{34}$$

where S satisfies the equation

$$2i\gamma^2 b_z S = \nabla^2 S + F\mathrm{v}_z \tag{35}$$

isotropic in the horizontal plane.

Now Eq. (28) can be written as follows:

$$\frac{1}{\mathrm{Pr}}\frac{\partial\nabla^2\mathrm{v}_z}{\partial t} = \nabla^4\mathrm{v}_z + \mathrm{Ra}\,\Delta_2 T - \frac{1}{2}\mathrm{Ha}^2|F|^2\Delta_2\mathrm{v}_z - 2\gamma^2\mathrm{Ha}^2\Delta_2\mathrm{Im}(F^*S), \tag{36}$$
$$\nabla^2 S - 2i\gamma^2 S = -F\mathrm{v}_z. \tag{37}$$

For normal perturbations $T, v_z, S \sim \cos kx$, the boundary conditions to (37) are

$$\frac{dS}{dz} + kS = 0 \tag{38}$$

at the upper boundary and

$$\frac{dS}{dz} - kS = 0 \tag{39}$$

at the lower one.

We shall restrict ourselves to the analysis of the onset of monotonous instability. We consider the perturbations of the z-component of the velocity v_z, temperature T and the function S, characterizing z-component of magnetic field in the following form:

$$v_z = V(z)\cos kx, \qquad T = \vartheta(z)\cos kx, \\ S = s(z)\cos kx, \tag{40}$$

where V, ϑ are real-valued quantities, and s is a complex-valued quantity. Hence, we obtain following system of ordinary differential equations:

$$\left(\frac{d^2}{dz^2} - k^2\right)^2 V - \mathrm{Ra}\,k^2\vartheta + \frac{1}{2}\mathrm{Ha}^2 k^2 |F|^2 V \\ + 2\gamma^2 \mathrm{Ha}^2 k^2 \mathrm{Im}(F^* s) = 0, \tag{41}$$

$$\left(\frac{d^2}{dz^2} - k^2\right)\vartheta + V = 0, \tag{42}$$

$$\left(\frac{d^2}{dz^2} - k^2\right)s - 2i\gamma^2 s = -FV \tag{43}$$

with a boundary conditions:

$$z = 0, 1: \qquad V = 0, \qquad \frac{d^2V}{dz^2} = 0, \qquad \vartheta = 0, \\ \frac{ds}{dz} = \pm ks \tag{44}$$

in the case of free boundaries and

$$z = 0, 1: \qquad V = 0, \qquad \frac{dV}{dz} = 0, \qquad \vartheta = 0, \\ \frac{ds}{dz} = \pm ks \tag{45}$$

in the case of rigid boundaries. This system was reduced to a system of ordinary differential equations of the first order and solved by a Runge–Kutta method of the 4th order.

The neutral curves for free and rigid boundaries for different values of γ are presented in Figs. 1–4. In Figs. 5–8, the dependencies of the critical Rayleigh number and the corresponding critical wave number on the parameter γ are plotted for the cases of free and rigid boundaries. As one can see, at small values of γ (nearly homogeneous magnetic field) the critical Rayleigh number is higher than in the absence of a magnetic field (the Rayleigh–Benard problem) (a magnetic field exerts the stabilizing effect). For large values of γ, when magnetic field just weakly penetrates into the fluid, the boundary of monotonous instability coincides with the instability boundary of the Rayleigh–Benard problem.

Within the intermediate range of γ, an interesting phenomenon has been observed: for some γ the critical Rayleigh number is less than that in the absence of a magnetic field and for large enough values of the Hartmann number it is even negative. This means that under strong enough magnetic fields convection is possible even in a layer heated from above. This phenomenon is illustrated by Figs. 9 and 10, where the dependencies of the Hartmann number on the wave number of neutral disturbances at zero Rayleigh number are plotted for the cases of free and rigid boundaries, respectively. The instability exists only at $k < k_*$. The critical Hartmann number tends to infinity, when $k \to k_* - 0$. For $k > k_*$, neutral disturbances are absent.

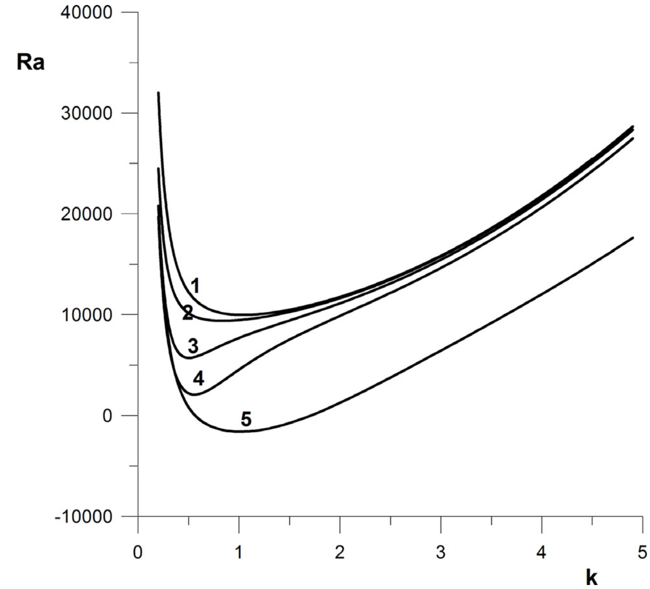

Fig. 1. Neutral curves for Ha = 40, $\gamma = 0,\ 0.5,\ 0.75,\ 1,\ 1.9$ (curves 1, 2, 3, 4, 5 respectively), (free boundaries).

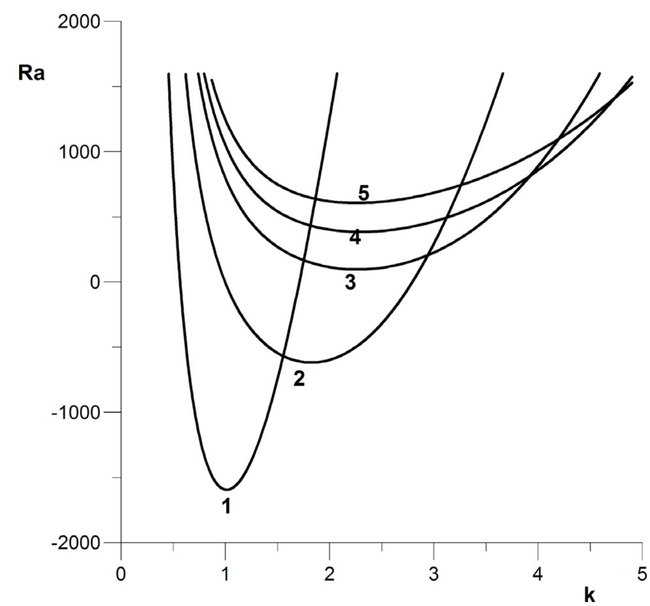

Fig. 2. Neutral curves for Ha = 40, $\gamma = 1.9,\ 3,\ 4,\ 5,\ 10$ (curves 1, 2, 3, 4, 5 respectively), (free boundaries).

So, we can conclude that a weakly inhomogeneous rotating magnetic field suppresses convection and, on the contrary, a strongly inhomogeneous field can intensify it. The intensification effect is maximal for a certain finite value of the parameter γ; at a further growth of γ this effect monotonously decreases and at very large values of γ it vanishes (skin-layer case).

The critical wavenumber in the case of a rapidly rotating inhomogeneous magnetic field is less than that in the absence of a magnetic field. Its decrease is maximal at a certain finite value of γ and vanishes at large γ (the case of weak penetration of magnetic field into the fluid).

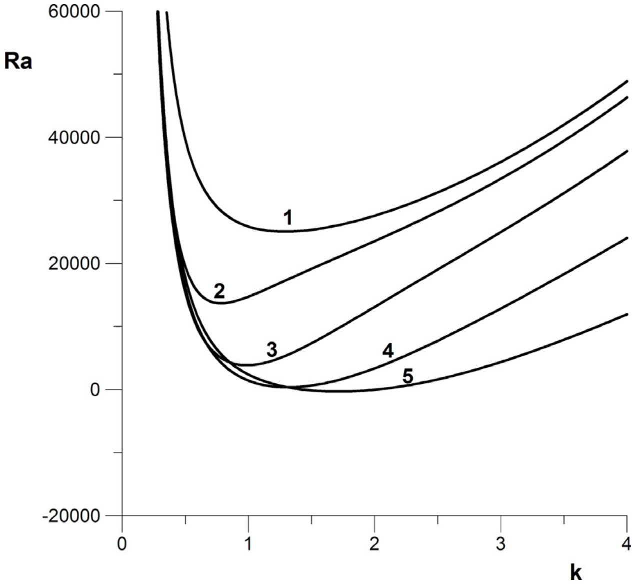

Fig. 3. Neutral curves for Ha = 60, $\gamma = 0,\ 1,\ 1.5,\ 2,\ 2.5$ (curves 1, 2, 3, 4, 5 respectively), (rigid boundaries).

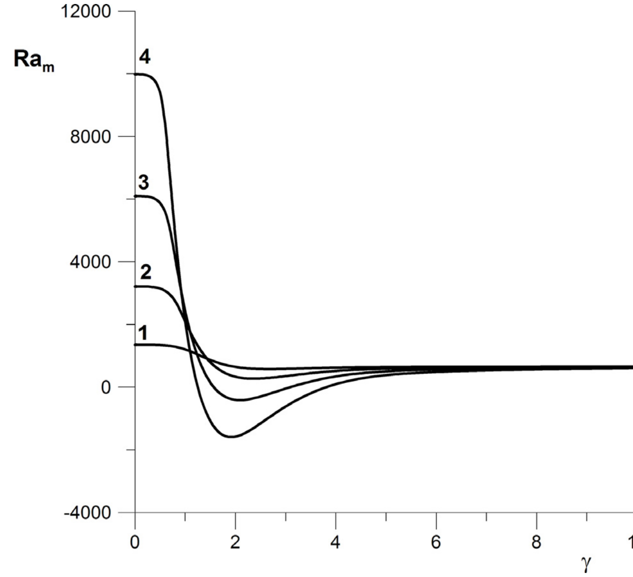

Fig. 5. Dependence of minimal critical Rayleigh number Ra_m on the parameter γ for Ha = 10, 20, 30, 40 (curves 1, 2, 3, 4 respectively), (free boundaries).

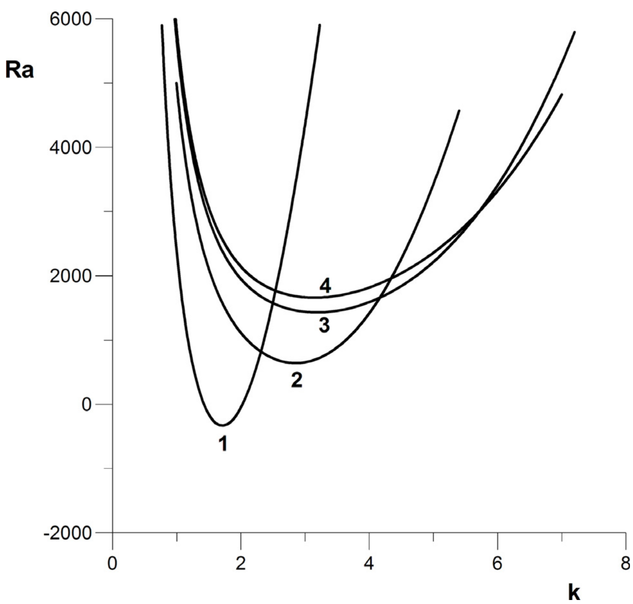

Fig. 4. Neutral curves for Ha = 60, $\gamma = 2.5,\ 4,\ 6, 10$ (curves 1, 2, 3, 4 respectively), (rigid boundaries).

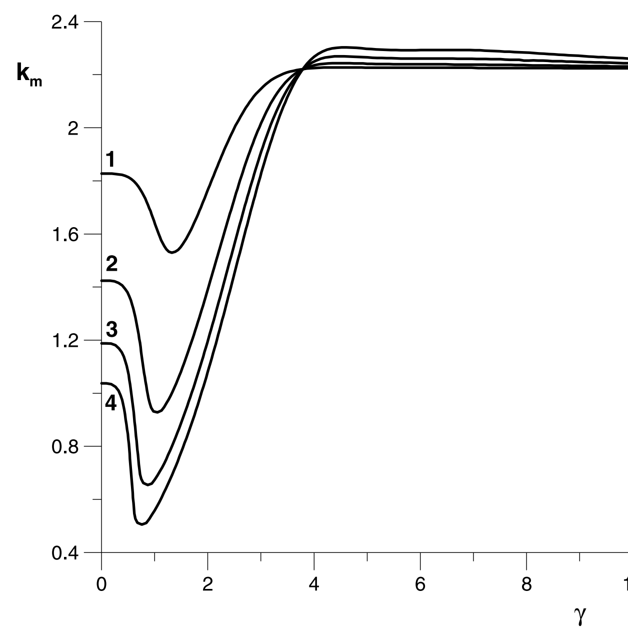

Fig. 6. Dependence of the wave number of most dangerous perturbations k_m on the parameter γ for Ha = 10, 20, 30, 40 (curves 1, 2, 3, 4 respectively), (free boundaries).

4.2. The case of finite frequency rotation

Now, we turn to consider the case of finite frequency for stress-free boundaries. Applying a double curl operation to Eq. (19) and projecting it onto the vertical direction, we get:

$$\frac{1}{\mathrm{Pr}}\frac{\partial \nabla^2 \mathrm{v}_z}{\partial t} = \nabla^4 \mathrm{v}_z + \mathrm{Ra}\,\Delta_2 T + \frac{\mathrm{Ha}^2\,\mathrm{Pr}_m}{\mathrm{Pr}}\left(\vec{B}_0 \cdot \nabla\nabla^2 B_z - \frac{\partial^2 \vec{B}_0}{\partial z^2}\cdot \nabla B_z\right), \tag{46}$$

$$\frac{\partial T}{\partial t} = \nabla^2 T + \mathrm{v}_z, \tag{47}$$

$$\frac{\partial B_z}{\partial t} = \frac{\mathrm{Pr}}{\mathrm{Pr}_m}\nabla^2 B_z + \vec{B}_0 \cdot \nabla \mathrm{v}_z. \tag{48}$$

We search for the disturbances of z-component of velocity, temperature and z-component of magnetic induction in the following form:

$$\mathrm{v}_z = W\ (z, t)\, e^{i(k_x x + k_y y)} + c.c.,$$
$$T = \vartheta\ (z, t)\, e^{\,i\,(k_x x + k_y y)} + c.c.,$$
$$B_z = B(z, t) e^{i(k_x x + k_y y)} + c.c.$$

After introduction of the notation $\varphi = \left(\frac{\partial^2}{\partial z^2} - k^2\right) W$, where $k^2 = k_x^2 + k_y^2$, the following system of equations is derived:

$$\frac{1}{\mathrm{Pr}}\frac{\partial \varphi}{\partial t} = \left(\frac{\partial^2}{\partial z^2} - k^2\right)\varphi - k^2 \mathrm{Ra}\,\vartheta + \frac{\mathrm{Ha}^2\,\mathrm{Pr}_m}{\mathrm{Pr}}\left(ikB_{0x}\left(\frac{\partial^2}{\partial z^2} - k^2\right)B - ik\frac{\partial^2 B_{0x}}{\partial z^2}B\right), \tag{49}$$

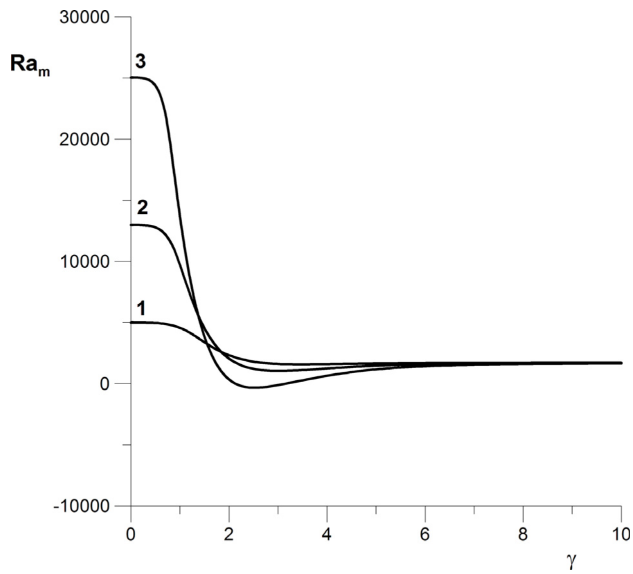

Fig. 7. Dependence of minimal critical Rayleigh number Ra_m on the parameter γ: for Ha = 20, 40, 60 (curves 1, 2, 3 respectively), (rigid boundaries).

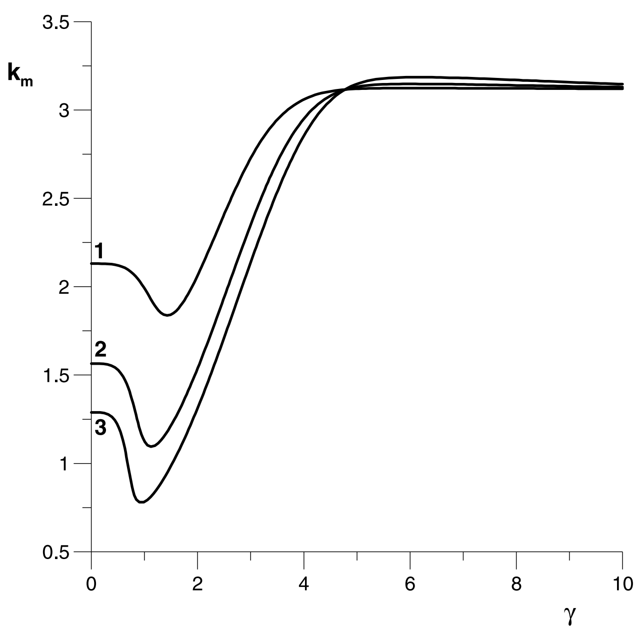

Fig. 8. Dependence of the wave number of most dangerous perturbations k_m on the parameter γ for Ha = 20, 40, 60 (curves 1, 2, 3 respectively), (rigid boundaries).

$$\frac{\partial \vartheta}{\partial t} = \nabla^2 \vartheta + W, \tag{50}$$

$$\frac{\partial B}{\partial t} = \frac{\mathrm{Pr}}{\mathrm{Pr}_m} \nabla^2 B + ikB_{0x} W, \tag{51}$$

$$\nabla^2 W = \varphi. \tag{52}$$

The boundary conditions for the velocity and temperature are:

$$z = 0, 1: \quad W = \frac{\partial^2 W}{\partial z^2} = \vartheta = 0. \tag{53}$$

The layer is surrounded by a dielectric solid. Based on the equation for the magnetic field in a solid $\nabla^2 \vec{B}_m = 0$ (magnetic permeabilities in solid and fluid are considered as a constant) and taking into account the continuity of perturbations on boundary,

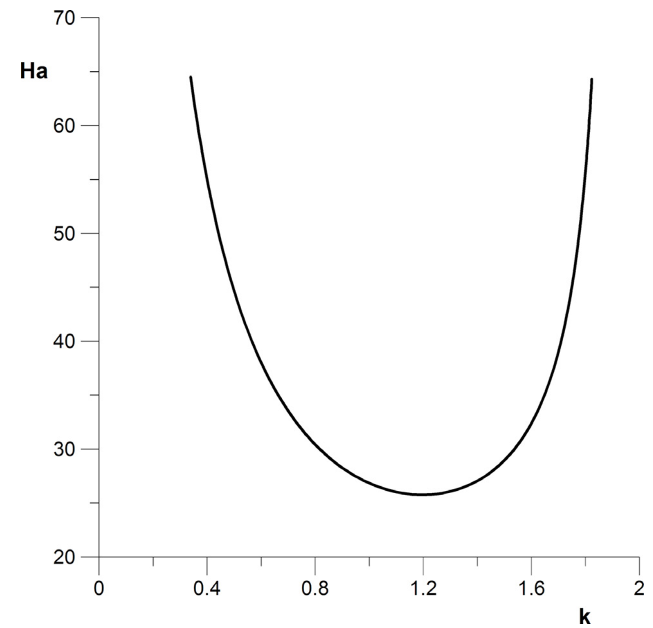

Fig. 9. Dependence of the Hartmann number Ha on the wave number k of neutral disturbances (free boundaries).

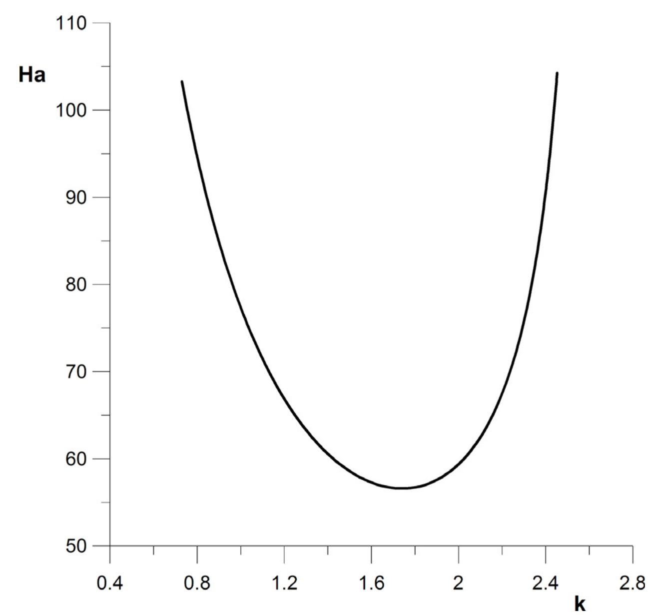

Fig. 10. Dependence of the Hartmann number Ha on the wave number k of neutral disturbances (rigid boundaries).

the following boundary conditions are obtained:

$$z = 0: \quad \frac{\partial B}{\partial z} = kB, \tag{54}$$

$$z = 1: \quad \frac{\partial B}{\partial z} = -kB. \tag{55}$$

The system of equations (49)–(52)with boundary conditions (53)–(55) was solved numerically by the finite difference method. The evolutionary equations were solved using an explicit scheme. The equation for W was solved by the scalar sweep method.

Fig. 11 shows the neutral curves at Pr = 0.0254 (mercury), Ha = 10, $\Omega = 0.1$ for some ranges of the inhomogeneity parameter γ. One can observe resonance branches on the neutral curves. In this figure, the branches related to synchronous solutions with

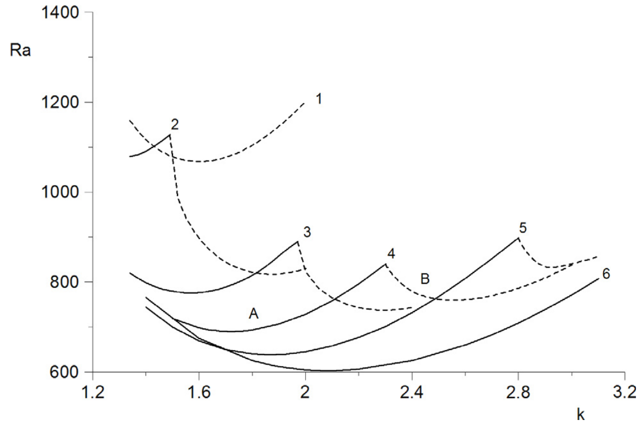

Fig. 11. Neutral curve for Pr = 0.0254, Ha = 10, $\Omega = 0.1$, $\gamma = 1, 1.5, 2, 2.25, 2.5, 3$ (curves 1, 2, 3, 4, 5, 6 respectively).

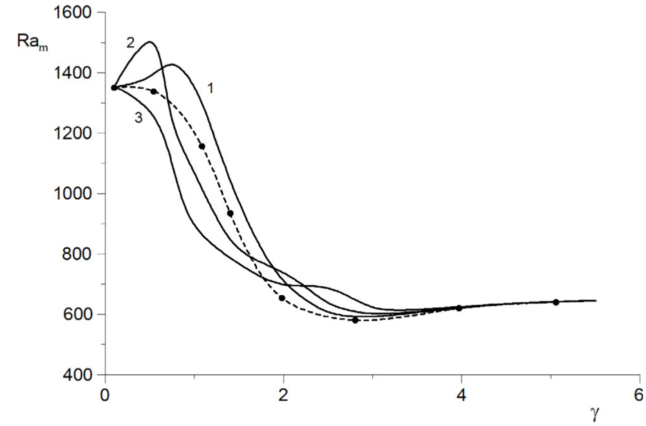

Fig. 15. Dependence of critical Rayleigh number Ra_m on inhomogeneity parameter γ for Pr = 0.0254, Ha = 10, $\Omega = 0.2, 0.1, 0.05$ (solid lines 1, 2, 3 respectively), $\Omega = 10$ (points) and for high-frequency limit (dashed line).

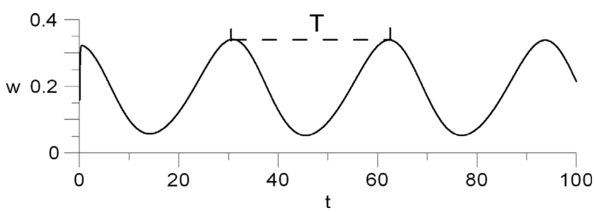

Fig. 12. The shape of oscillations of $w(t)$ at $\gamma = 2.25$, $k = 1.8$, Ra = 693.

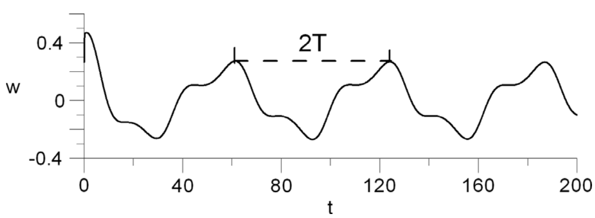

Fig. 13. The shape of oscillations of $w(t)$ at $\gamma = 2.25$, $k = 2.6$, Ra = 760.

period τ are shown by solid lines, and the branches related to subharmonic solutions with a period 2τ are shown by a dashed line. In Figs. 12 and 13 the shapes of oscillations are given for synchronous and subharmonic modes. Fig. 14 presents the structure of critical perturbations of the velocity and magnetic field.

In Figs. 15 and 16, the dependencies of critical Rayleigh number Ra_m and critical wavenumber number k_m on the inhomogeneity parameter γ are presented for several values of dimensionless field rotation frequency and for high-frequency limit. One can see a significant stabilization of motionless state for small values of inhomogeneity parameter as in the case of high-frequency limit. A decrease in frequencies first leads to stabilization, but then the behavior becomes more complex. For some values of the inhomogeneity parameter (not very large), the stabilization effect is stronger than in the high-frequency case, and for other values the effect is weaker, but, in general, the stabilizing effect of a rotating field remains unchanged. For inhomogeneity parameter values, where in the high-frequency case there is destabilization, this effect becomes weaker, and at some values of γ at sufficiently low frequencies it is replaced by stabilization. The destabilization domain is shifted to the region of large inhomogeneity parameter values. The dependence of a critical wavenumber on the inhomogeneity parameter for sufficiently high frequencies is qualitatively the same as for the high-frequency limit. At decreasing frequency with appearance of resonance branches on the neutral curve, the additional extrema appear for the function $k_m(\gamma)$.

In Figs. 17 and 18, the dependencies of the critical Rayleigh number Ra and wavenumber on the inhomogeneity parameter γ are shown for Pr = 0.0254, Ha = 40 and several values of the dimensionless magnetic field rotation frequency and for high-frequency limit. As one can see from Fig. 17, in sufficiently strong fields in the destabilization region a reduction in the critical Rayleigh number to negative values is possible. For high-frequency case, this has been discussed in the previous section. However, in the case of low frequencies this effect disappears due to a reduction in the destabilization effect.

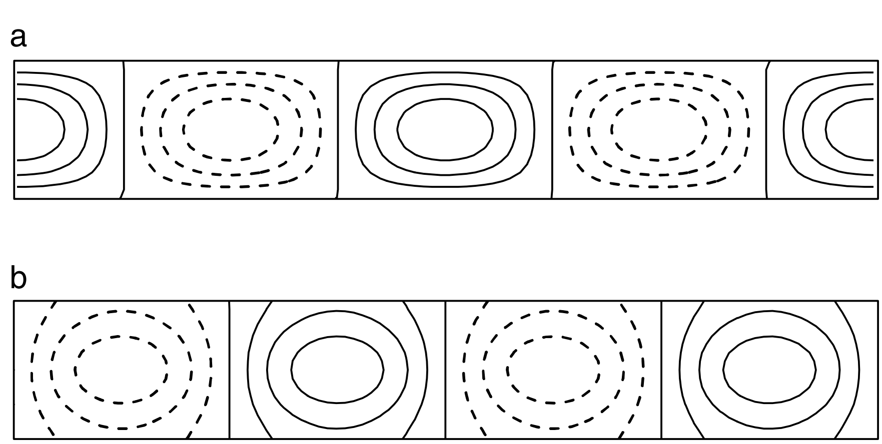

Fig. 14. The structure of critical perturbations of velocity w (a) and magnetic field H(b) for Pr = 0.0254, Ha = 10, $\Omega = 0.1$, $\gamma = 3$, $k = 2.0$, Ra = 605.

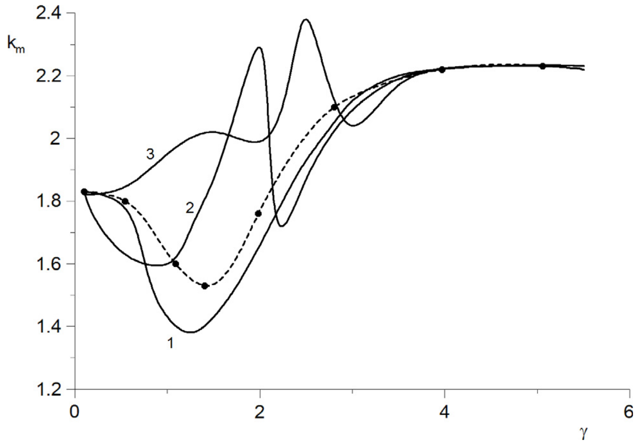

Fig. 16. Dependence of the wave number of most dangerous perturbations k_m on inhomogeneity parameter γ for Pr = 0.0254, Ha = 10, Ω = 0.2, 0.1, 0.05 (solid lines 1, 2, 3 respectively), Ω = 10 (points) and for high-frequency limit (dashed line).

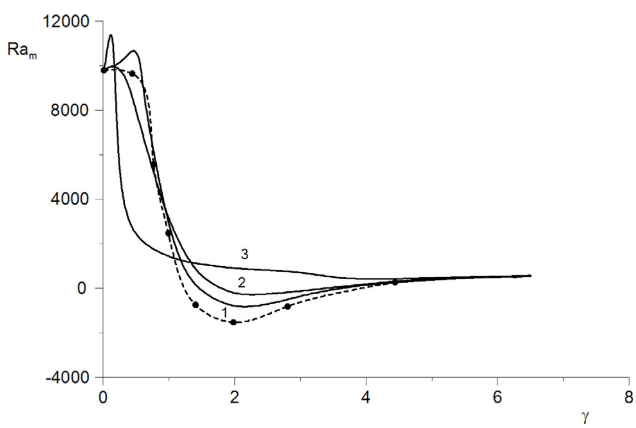

Fig. 17. Dependence of critical Rayleigh number Ra_m on inhomogeneity parameter γ for Pr = 0.0254, Ha = 40, Ω = 1, 0.5, 0.05 (solid lines 1, 2, 3 respectively), Ω = 100 (points) and for high-frequency limit (dashed line).

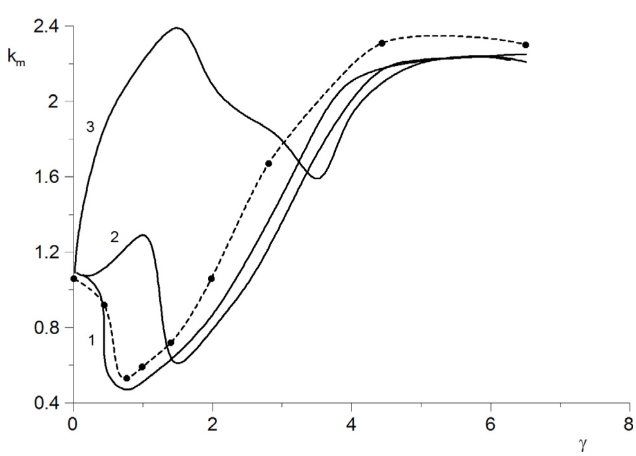

Fig. 18. Dependence of the wave number of most dangerous perturbations k_m on inhomogeneity parameter γ for Pr = 0.0254, Ha = 40, Ω = 1, 0.5, 0.05 (solid lines 1, 2, 3 respectively), Ω = 100 (points) and for high-frequency limit (dashed line).

5. Discussion of results

For very high rotation frequencies the magnetic field can no longer be considered as homogeneous. The analysis made in our paper takes into account its inhomogeneity. It is obvious that in the limit case of large rotation frequencies, the magnetic field does not penetrate into the bulk of conducting fluid: its intensity essentially reduces outside the skin-layers at fluid surfaces. As a result, the magnetic field no longer affects the stability of the conductive state. The calculations show that the reduction of the magnetic field effect occurs in a non-monotonous way. Moreover, at some values of the inhomogeneity parameter the conductive state turns out to be unstable in the absence of heating and even under heating from above, i.e. for the potentially stable stratification of a fluid. This demonstrates the existence of a specific instability mechanism related to the magnetic field inhomogeneity.

At first glance the onset of convection at negative Rayleigh numbers in the presence of a magnetic field seems to be surprising. However, as soon as the inhomogeneity of the magnetic field has been recognized the source of energy, it becomes evident that we are dealing with the magnetic instability in the presence of a stably stratified density field. Almost similar situation has been encountered in the case of convection in a rotating spherical fluid shell in the presence of a purely azimuthal magnetic field [27]. With increasing magnetic field strength, the negative critical Rayleigh numbers have been obtained for convection modes characterized by low azimuthal wave numbers, such as $m = 1$ and $m = 2$. A further well-known example of the hydrodynamically stable state that becomes unstable in the presence of a magnetic field arises in the Couette–Taylor problem. The cylindrically symmetric state of an azimuthal flow becomes unstable in the presence of a axial magnetic field even when the Rayleigh criterion for instability is not satisfied (Velikhov, 1959; Chandrasekhar, 1961; Balbus & Hawley, 1991) [28,4,29]. Since the magnetic field is homogeneous in this case, it just facilitates the instability, but does not provide the energy of the growing disturbances, which is still derived from the basic shear flow. This magneto-rotational instability plays an important role in the origin of turbulence in accretion disks [29].

Resonance branches were obtained at finite values of the rotation frequency. The stabilization of the motionless state in the magnetic field may become stronger or weaker depending on the parameter values. The destabilization effect is attenuated (at least in the examined cases). The dependence of the critical wavenumber appears more complex than in the high-frequency case.

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