Spiral-defect chaos: Swift-Hohenberg model versus Boussinesq equations

Rainer Schmitz,[1,*] Werner Pesch,[2] and Walter Zimmermann[1,3]
[1]*Institut für Festkörperforschung, Forschungszentrum Jülich, D-52425 Jülich, Germany*
[2]*Theoretische Physik II, Universität Bayreuth, D-95440 Bayreuth, Germany*
[3]*Theoretische Physik, Universität des Saarlandes, D-66041, Saarbrücken, Germany*
(Received 19 July 2001; published 7 March 2002)

Spiral-defect chaos (SDC) in Rayleigh-Bénard convection is a well-established spatio-temporal complex pattern, which competes with stationary rolls near the onset of convection. The characteristic properties of SDC are accurately described on the basis of the standard three-dimensional Boussinesq equations. As a much simpler and attractive two-dimensional model for SDC generalized Swift-Hohenberg (SH) equations have been extensively used in the literature from the early beginning. Here, we show that the description of SDC by SH models has to be considered with care, especially regarding its long-time dynamics. For parameters used in previous SH simulations, SDC occurs only as a transient in contrast to the experiments and the rigorous solutions of the Boussinesq equations. The small-scale structure of the vorticity field at the spiral cores, which might be crucial for persistent SDC, is presumably not perfectly captured in the SH model.

DOI: 10.1103/PhysRevE.65.037302 PACS number(s): 47.20.Bp, 05.45.−a, 47.27.Te

Convection in a horizontal fluid layer heated from below, known as Rayleigh-Bénard convection, is one of the best-studied examples of pattern forming systems [1–3]. At threshold, convection rolls bifurcate and remain stable in a fairly wide parameter range, coined as the Busse-Balloon [1,2]. Thus, the recent observation of spiral-defect chaos (SDC) in a parameter regime where it competes with rolls was rather surprising [4,5]. The complex spatio-temporal dynamics of SDC involves rotating spirals, targets, dislocations, etc. Most of the characteristic properties of SDC are well reproduced in high-precision *ab initio* solutions of the standard Boussinesq equations [6–8] in three spatial dimensions. According to the experiments and the numerical solutions, SDC is a robust generic state of thermal convection observed in rectangular, square, and circular cells, as well (see, e.g., [3–10]).

Our general understanding of the universal aspects of pattern formation has been significantly promoted by the analysis of two-dimensional models such as the various types of Ginzburg-Landau and Swift-Hohenberg equations [1,11,12]. This applies also to the generalized Swift-Hohenberg (SH) equations [13,14] which include mean-flow effects important at low Prandtl numbers in thermal convection. The numerical simulation of SH models has provided important insight into the underlying mechanism of SDC [15–18]. Nevertheless, model equations imply approximations and their possible limitations have to be examined carefully. Here, we show that the long-time dynamics of SDC in terms of the generalized SH equations might be problematic.

In the following, we discuss simulations of SDC in a set of widely used generalized SH equations, which couple two real fields $\psi(\mathbf{r},t)$ and $\zeta(\mathbf{r},t)$ (see, e.g., [15,16])

$$[\partial_t + g_m \mathbf{U}\cdot\boldsymbol{\nabla}]\psi = [\varepsilon - (1+\Delta)^2]\psi - \psi^3, \tag{1a}$$

$$[\tau_\zeta \partial_t - \mathcal{P}(\eta\nabla^2 - c^2)]\Delta\zeta = [(\partial_y\psi)\partial_x - (\partial_x\psi)\partial_y]\Delta\psi. \tag{1b}$$

$\psi(\mathbf{r},t)$ describes the planar spatial variations of convection patterns (e.g., the temperature field), which consist locally of convection-roll patches. $\zeta(\mathbf{r},t)$ is a velocity potential determining the mean flow $\mathbf{U} = (\partial_y\zeta, -\partial_x\zeta)$. The control parameter $\varepsilon = 2.78\,(\Delta T - \Delta T_c)/\Delta T_c$ serves as a dimensionless measure for the applied temperature difference ΔT across the fluid layer [19]. The time is scaled in such a way that a time lapse of $t=5$ in Eqs. (1) corresponds to the common vertical diffusion time t_v, which is about a few seconds in experiments.

Any curvature of the rolls produces a vertical vorticity field $-\Delta\zeta(\mathbf{r},t)$ which increases with decreasing Prandtl number $\mathcal{P}$ according to Eq. (1b). In contrast to the claims expressed in several papers by Xi and Gunton (see, e.g., [20]), only the dominant term $\sim c^2$ on the left-hand side of Eq. (1b) may be directly traced back to the Boussinesq equations. The two other terms $\propto \tau_\zeta, \eta$, respectively, are in principle phenomenological, as discussed in some detail in Ref. [17]. In Eq. (1a), the relevance of $\zeta(\mathbf{r},t)$ is controlled by the coupling constant g_m. The value of g_m may be found to be $g_m = 12.2$ for $c^2 = 2$ and $\mathcal{P} = 1$ by comparison with the known zigzag stability boundary of convection rolls [21].

The coupling to the mean flow, which becomes more important either at small $\mathcal{P}$ or large g_m is crucial for persistent SDC. In the limit of large Prandtl numbers $\mathcal{P}$, where ζ is hardly excited, the dynamics of ψ becomes purely relaxational and approaches a low dimensional stationary state of the corresponding Lyapunov functional [1,12]. Note, however, that any strongly disordered pattern before it equilibrates generates virtually instantaneously a strong, long-range mean-flow $\mathbf{U}$ according to Eq. (1b) and may thus easily lead to a transient SDC-like dynamics.

In our numerical solutions of Eqs. (1), we have chosen the same set of parameters as in the previous works [15,16,20], namely, $c^2 = 2$, $g_m = 50$, $\tau_\zeta = \eta = \mathcal{P} = 1$, $\varepsilon = 0.7$. Mostly, we

*Present address: AGFA Gaevaert AG, P.O. Box 100160, D-51301 Leverkusen, Germany.

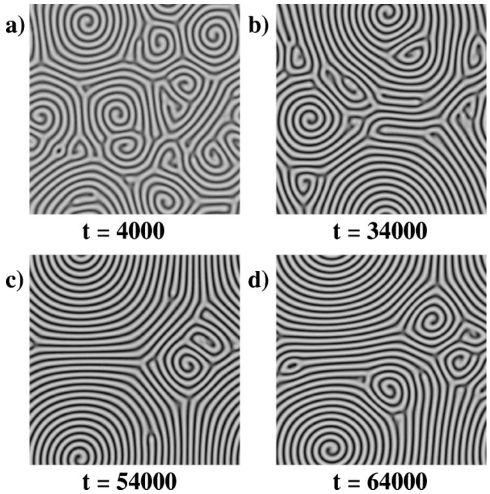

FIG. 1. The field $\psi(\mathbf{r},t)$ (1) is plotted at increasing times t after starting with random initial conditions at $t=0$. The parameters are $\Gamma=32$ (aspect ratio), $\varepsilon=0.7$, $g_m=50$, $c^2=2$, $\eta=1$, $\mathcal{P}=1$, and $\tau_\zeta=1$.

consider an aspect ratio of $\Gamma=L/2d=32$, where L denotes the lateral extension of the cell and d its thickness. At first, we have performed simulations in a square domain with periodic boundary conditions in order to avoid an artificial bias from the sides. Starting from random initial conditions yields a typical snapshot as shown in Fig. 1(a) at $800t_v$. This pattern compares well with planforms already shown in Refs. [15,16] at the same time lapse. It resembles also the characteristic SDC snapshots observed persistently in experiments [4,5] or during numerical solutions of the fundamental Boussinesq equations [6].

However, when continuing the runs beyond $8000t_v$ the scenario changes qualitatively and the pattern coarsens towards a ''big spiral'' as shown in Figs. 1(c) and 1(d), which rotates about a slowly migrating center. Only at the boundaries of the big spiral one finds remnants of the previous persistent generation and annihilation of small spirals. For $\mathcal{P}\approx 1$, the coarsening to big spirals is neither observed in experiments nor during simulations of the Boussinesq equations.

The transient behavior of SDC followed by a coarsening to a big spiral reminds us of recent experiments at $\mathcal{P}=4$ [24]. After a sudden quench, strongly disordered pattern developed, which due to the strong vorticity field, led to the SDC transient. Afterwards, SDC coarsened to a big spiral as well, which eventually disintegrated after a long time towards a stationary pattern. Apparently for $\mathcal{P}=4$, the vorticity field is too weak to sustain SDC.

Unlike the experiments and the solutions of the Boussinesq equations, the SDC attractor in the SH simulations shows a remarkable sensitivity to the boundary conditions. This feature becomes evident when Eqs. (1) are simulated on a circular domain as in the previous work in Ref. [15]. Initially, we observe a similar coarsening process as in the case of the square domain leading to a big spiral about the cell center that is rather long living (on the average $500–1000t_v$). Possibly because of the focus instability [25,26] the spiral core moves off center and the spiral arms may be compressed and react in a sudden process by the generation of dislocation pairs, inevitably associated with a strong vorticity field. The dislocation tips wind up in a dynamics, which has been loosely described as ''invasive chaos'' by Cross and Tu [16,17]. During that period, one observes SDC that coarsens again to a quite big spiral, which becomes again unstable and so on. The periodic dynamics due to the generation of dislocations in compressed roll patches is typical for circular cells and has been described in other comparable situations as well (see, e.g., [27]).

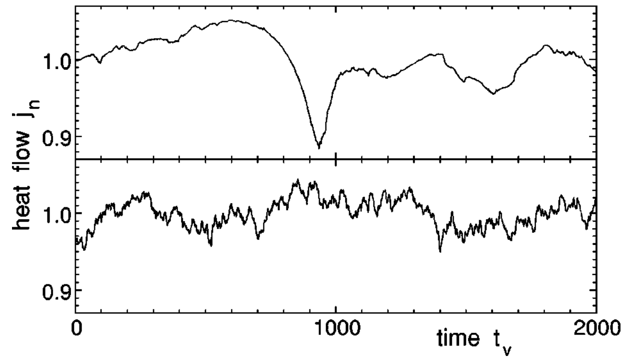

FIG. 2. The normalized convective heat flux $j_n(t)$ [see Eq. (2)] is shown for the SH model (upper part) with the same parameters as in Fig. 1 and for the Boussinesq equations (see, e.g., [6]) (lower part).

The difference between the latter scenario of ''intermittent SDC'' and persistent generic SDC is obvious from Fig. 2. Here, we compare in a circular geometry the normalized convective heat current $j_n(t)$ obtained from simulations of Eqs. (1) with the same quantity in the full Boussinesq equations [6]. For the SH model, $j_n(t)$ is given as

$$j_n(t)=\frac{j(t)}{\langle j(t)\rangle}, \tag{2}$$

with the heat current

$$j(t)=\frac{1}{S}\int_S d^2\mathbf{r}\,\psi^2(\mathbf{r},t). \tag{3}$$

$\langle \dots \rangle$ denotes the time average and S the area of the cell. In both cases, the heat current is only shown for a small representative time window taken out of a very long runs that lasted up to $t_v=40\,000$. In the upper panel, we see rather rare but strong events, when the coarsened big spirals breaks up at the downward spike and cell-filling SDC appears for a short time. In contrast, the heat current in the Boussinesq case (lower panel) shows only moderate fluctuations corresponding to the typical small scale dynamics of SDC.

We have convinced ourselves that also for larger $\Gamma=64$ while the other parameters are chosen as in Fig. 1, the dominant coarsening to big spirals cannot be effectively suppressed. However, they become unstable from time to time and in analogy to Fig. 2, intermediate SDC appears.

With respect to some deficiencies of the SH model alluded to above, we have several speculations. Already when

Manneville introduced the generalized SH equations in Ref. [13], he discussed in detail the intricate roll of the mean flow and some very long transients before his simulations settled down to a steady attractor. The fact that the Busse balloon is not correctly reproduced by the generalized SH model [6,23,20] and that even transient SDC requires a four times larger g_m value than the theoretical one (see above and [20]) might be of minor importance. From a more general point of view, the generalized SH equations are based on a long wave-length approximation for $\zeta(\mathbf{r},t)$. Therefore, the pronounced short-scale structures in the vorticity field, which exist, for instance, at a spiral core (see, e.g., Fig. 18 in [3]), are not systematically captured by construction. They might be responsible for a permanent ''stirring'' of the real system and for keeping a persistent weak turbulence alive.

In conclusion, for many purposes, generalized SH models are certainly very valuable tools to study the SDC scenario, even if it would exist only as a long transient. However, our investigations shed some light on the general problem of understanding the long-time behavior of hydrodynamic systems by using SH models. Accordingly, their application to coarsening studies [28] or to the analysis of statistical properties of SDC [20] might be questionable.

[1] M.C. Cross and P.C. Hohenberg, Rev. Mod. Phys. **65**, 851 (1993).

[2] F.H. Busse, in *Convection: Plate Tectonics and Global Dynamics*, edited by W. R. Peltier (Gordon and Breach, Montreux, 1989), pp. 23–95.

[3] E. Bodenschatz, W. Pesch, and G. Ahlers, Annu. Rev. Fluid Mech. **32**, 709 (2000).

[4] S.W. Morris, E. Bodenschatz, D.S. Cannell, and G. Ahlers, Phys. Rev. Lett. **71**, 2026 (1993).

[5] M. Assenheimer and V. Steinberg, Nature (London) **367**, 345 (1994).

[6] W. Decker, W. Pesch, and A. Weber, Phys. Rev. Lett. **73**, 648 (1994).

[7] W. Pesch, Chaos **6**, 348 (1996).

[8] D.A. Egolf, I.V. Melnikov, W. Pesch, and R.E. Ecke, Nature (London) **404**, 733 (2000).

[9] S.W. Morris, E. Bodenschatz, D. Cannel, and G. Ahlers, Physica D **97**, 164 (1996).

[10] B.P.R. Cakmur, D. Egolf, B.B. Plapp, and E. Bodenschatz, Phys. Rev. Lett. **79**, 1853 (1997).

[11] A.C. Newell, T. Passot, and J. Lega, Annu. Rev. Fluid Mech. **25**, 399 (1992).

[12] J. Swift and P.C. Hohenberg, Phys. Rev. A **15**, 319 (1977).

[13] P. Manneville, J. Phys. (France) Lett. **44**, L903 (1983).

[14] H.S. Greenside, M.C. Cross, and W.M. Coughran, Phys. Rev. Lett. **60**, 2269 (1988).

[15] H.W. Xi, J.D. Gunton, and J. Vinals, Phys. Rev. Lett. **71**, 2030 (1993).

[16] M.C. Cross and Y. Tu, Phys. Rev. Lett. **75**, 834 (1995).

[17] M.C. Cross, Physica D **97**, 65 (1996).

[18] M. Bestehorn, M. Fantz, R. Friedrich, and H. Haken, Phys. Lett. A **174**, 48 (1993).

[19] The detailed relations between the model parameters and the corresponding physical quantities have been addressed in Refs. [1,6] and, in particular, in Ref. [20].

[20] H.W. Xi and J.D. Gunton, Phys. Rev. E **52**, 4963 (1995); X.J. Li, H.W. Xi, and J.D. Gunton, *ibid.* **57**, 1705 (1998).

[21] The coupling constant g_m depends weakly on the Prandtl number $\mathcal{P}$ in the vicinity of 1. Its relevance to the zigzag instability is for instance discussed in [22,23].

[22] M. Cross, Phys. Rev. A **27**, 490 (1983).

[23] W. Decker and W. Pesch, J. Phys. II **4**, 419 (1994).

[24] K.M. Bajaj, D. Cannell, and G. Ahlers, Phys. Rev. E **55**, R4869 (1997).

[25] A.C. Newell, T. Passot, and M. Souli, Phys. Rev. Lett. **64**, 2378 (1990).

[26] Y. Hu, R.E. Ecke, and G. Ahlers, Phys. Rev. Lett. **72**, 2191 (1994).

[27] V. Croquette, Contemp. Phys. **30**, 153 (1989).

[28] M.C. Cross and D.I. Meiron, Phys. Rev. Lett. **75**, 2152 (1995).