Homologous onset of double layer convection

F. H. Busse* and M. Petry
Physikalisches Institut, Universität Bayreuth, D-95440 Bayreuth, Germany
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The onset of convection in two superimposed fluid layers of the same height is considered. It is found that the neutral curve for $R(a)$ for the onset Rayleigh number R in dependence on the wave number a is an invariant of a multidimensional parameter space of property ratios of the system even though the corresponding convection solutions may vary strongly with these property ratios. For each neutral curve $R(a)$ two manifolds of solutions are found one of which can be understood on the basis of symmetry properties of the system, while the other does not exhibit simple symmetry features. In particular the neutral curves $R(a)$ for various single Rayleigh-Bénard convection layers are shown to correspond to two two-dimensional manifolds of solutions. Analytical expressions for the latter are derived in the case of outer stress-free boundary conditions.

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I. INTRODUCTION

Convection in two superimposed horizontal layers of immiscible fluids has received much attention because of its unusual dynamical features. Originally this problem was considered in the geophysical context [1,2], since there exists evidence that convection in the Earth's mantle may occur in two layers separated by an interface at 660 km depth. Later an oscillatory mode of convection was found [3] and the possibility of resonances between convection flows in the two layers has been studied [4].

While most of the special features of double layer convection have been reasonably well understood, the large dimension of the parameter space of the problem inhibits a general overview of the properties of this dynamical system. In addition to the Rayleigh number and the Prandtl number which characterize convection in a single layer, many more dimensionless parameters must be considered in double layer convection, since the ratio of any material property of the liquids gives rise to a new dimensionless parameter.

In this paper we want to draw attention to relationships which are invariant of two-dimensional subspaces in the space of property ratios. In particular we shall demonstrate that the same neutral curve $R(a)$ for the onset of convection holds for two two-parametric manifolds of convection solutions even though the functional dependences of the convection motion and the associated deviation θ of the temperature from the static distribution vary dramatically. Such kinds of homologous bifurcations are a rather unexpected phenomenon and, to our knowledge, have not been found before. In the following we shall start with a formulation of the mathematical problem of double layer convection in the case when both layers have the same height. In Sec. III we shall consider the more realistic case of rigid outer boundaries of the double layer and in Sec. IV the assumption of stress-free outer boundaries will permit an analytical treatment of the problem. In both sections the role of the adjoint linear problem will be discussed. The paper closes with some concluding remarks in Sec. V.

*busse@uni-bayreuth.de

II. MATHEMATICAL FORMULATION

We consider a horizontal layer of fluid of thickness d superimposed onto another layer of the same thickness. The two liquids are immiscible. The temperatures T_2 and T_1 at the lower and at the upper boundaries, respectively, of the double layer are fixed. The temperature T_0 of the interface in the case of the static state of pure conduction is given by

$$T_0 = \frac{\lambda^* T_1 + \lambda T_2}{\lambda^* + \lambda}, \tag{1}$$

where $\lambda(\lambda^*)$ denotes the thermal conductivity of the lower (upper) layer. Using d as length scale, d^2/κ as time scale where κ is the thermal diffusivity of the lower liquid, and $(T_2 - T_0)$ as scale of the temperature we obtain the dimensionless equations

$$\frac{1}{P}(\partial_t + \mathbf{v}^* \cdot \nabla)\mathbf{v}^* = -\nabla \pi^* + \gamma_0 R \theta^* \mathbf{k} + \nu_0 \nabla^2 \mathbf{v}^*, \tag{2a}$$

$$\nabla \cdot \mathbf{v}^* = 0, \tag{2b}$$

$$(\partial_t + \mathbf{v}^* \cdot \nabla)\theta^* = \frac{1}{\lambda_0}\mathbf{v}^* \cdot \mathbf{k} + \kappa_0 \nabla^2 \theta^*, \tag{2c}$$

describing convection in the upper layer. The symbols γ_0, μ_0, ν_0, λ_0, and κ_0 denote the ratios of thermal expansivity, dynamic viscosity, kinematic viscosity, thermal conductivity, and thermal diffusivity, respectively, between the upper and the lower layer,

$$\gamma_0 = \frac{\gamma^*}{\gamma}, \quad \mu_0 = \frac{\mu^*}{\mu}, \quad \nu_0 = \frac{\nu^*}{\nu},$$

$$\lambda_0 = \frac{\lambda^*}{\lambda}, \quad \kappa_0 = \frac{\kappa^*}{\kappa}. \tag{3}$$

Equation (2) thus applies to the lower layer if γ_0, ν_0, λ_0, and κ_0 are replaced by unity and the stars at the variables are dropped. The variables π and π^* in general involve the density ratio of the two layers, but they do not enter the analysis considered in the following. The Rayleigh number R and the Prandtl number P are defined by

$$R = \frac{\gamma g (T_2 - T_0) d^3}{\nu\kappa}, \quad P = \frac{\nu}{\kappa}, \tag{4}$$

where g is the acceleration of gravity. Since we consider the conditions for onset of convection we drop the nonlinear advection terms in the basic equations. We also restrict the attention to the monotonous onset of convection which is usually preferred [5]. The time derivatives can thus be neglected.

Using a Cartesian system of coordinates (x,y,z) with the z coordinate in the direction of $\mathbf{k}$ we obtain from the z component of the $(\mathrm{curl})^2$ of Eq. (2a) and its counterpart for the lower layer

$$\nabla^4 w^* + R\frac{\gamma_0}{\nu_0}\Delta_2\theta^* = 0, \tag{5a}$$

$$\nabla^4 w + R\Delta_2\theta = 0, \tag{5b}$$

where w^* and w denote the z components of the velocity vectors $\mathbf{v}^*$ and $\mathbf{v}$ and Δ_2 is the two-dimensional Laplacian, $\Delta_2=\partial_{xx}^2+\partial_{yy}^2$. The corresponding versions of the heat equations are

$$\nabla^2\theta^* + \frac{1}{\lambda_0\kappa_0}w^* = 0, \tag{6a}$$

$$\nabla^2\theta + w = 0. \tag{6b}$$

Through the elimination of θ^* and θ we finally obtain

$$\left[\nabla^6 - R\frac{\gamma_0}{\nu_0\lambda_0\kappa_0}\Delta_2\right]w^* = 0, \tag{7a}$$

$$[\nabla^6 - R\Delta_2]w = 0. \tag{7b}$$

Usually the case is of interest when the Rayleigh numbers in the two layers are not very different. We shall consider here the special case when the Rayleigh numbers in the upper and lower layers are equal; i.e., we shall assume

$$\frac{\gamma_0}{\nu_0\lambda_0\kappa_0} = 1. \tag{8}$$

Since without loss of generality $\Delta_2 w^*=-a^2w^*$, $\Delta_2 w=-a^2w$ can be presumed, the x,y dependence can be separated from the z dependence, $w^*=f^*(z)g(x,y)$, $w=f(z)g(x,y)$, where $f(z)$ and $f^*(z)$ obey the same equation

$$[\Delta^3 + a^2R]f^{(*)} = 0, \tag{9}$$

where the definition $\Delta \equiv d^2/dz^2 - a^2$ has been introduced.

At the outer boundaries either stress-free conditions

$$f = f'' = \Delta^2 f = 0 \ \text{ at } \ z=-1,$$

$$f^* = f^{*\prime\prime} = \Delta^2 f^* = 0 \ \text{ at } \ z=1, \tag{10}$$

or no-slip conditions

$$f = f' = \Delta^2 f = 0 \ \text{ at } \ z=-1,$$

$$f^* = f^{*\prime} = \Delta^2 f^* = 0 \ \text{ at } \ z=1 \tag{11}$$

will be assumed. The condition that θ vanishes at the outer boundaries has also been taken into account in conditions (10) and (11). At the interface, $z=0$, distortions and effects due to the temperature dependence of surface tension will be neglected. The vertical velocity thus vanishes at the interface while the tangential velocities and the temperatures are equal on both sides,

$$\Delta^2 f - \Delta^2 f^*\frac{1}{\lambda_0\kappa_0} = f = f^* = f' - f^{*\prime} = 0 \ \text{ at } \ z=0. \tag{12}$$

The continuity of the heat flux and of the tangential stress require

$$\kappa_0\Delta^2 f' - \Delta^2 f^{*\prime} = f'' - \mu_0 f^{*\prime\prime} = 0 \ \text{ at } \ z=0, \tag{13}$$

where condition (8) has been used.

For later reference we like to mention the adjoint problem to the one just formulated. The solutions $\hat{f}(z)$ and $\hat{f}^*(z)$ of the adjoint problem obey the same Eq. (9) but different boundary conditions. In the case of stress-free outer boundary conditions (10) these remain unchanged, but the no-slip outer boundary conditions (11) are replaced by

$$\hat{f} = \Delta\hat{f} = \Delta\hat{f}' = 0 \ \text{ at } \ z=-1,$$

$$\hat{f}^* = \Delta\hat{f}^* = \Delta\hat{f}^{*\prime} = 0 \ \text{ at } \ z=1. \tag{14}$$

The conditions at the interface $z=0$ for the adjoint problem are given by

$$\hat{f} - \hat{f}^*\kappa_0 = \hat{f}' - \hat{f}^{*\prime}\lambda_0\kappa_0 = \Delta\hat{f} = \Delta\hat{f}^* = \Delta\hat{f}' - \Delta\hat{f}^{*\prime}/\mu_0$$
$$= \Delta^2\hat{f} - \Delta^2\hat{f}^* = 0 \ \text{ at } \ z=0. \tag{15}$$

In general the adjoint problem does not describe a physically realistic system.

In the following we wish to demonstrate the phenomenon of homologous bifurcation first in the case [Eq. (11)] of no-slip outer boundaries. Here numerical solutions of Eq. (9) will be employed which have been obtained through Runge-Kutta-type integrations. In order to gain more insight into the mathematical structure of the problem, we shall consider in Sec. IV analytical solutions of Eq. (9) with stress-free conditions (10).

III. RESULTS FOR NO-SLIP OUTER BOUNDARIES

A. Simple solutions

There are two special cases in which simple solutions of the two equations (9) together with boundary conditions (11)–(13) can be obtained. For $\lambda_0=\kappa_0=\mu_0=1$ either solutions f and f^* that are antisymmetric in z or solutions f and f^* that are symmetric in z can be obtained. In the former case one speaks of viscously coupled convection layers, while one refers to the latter case as thermally coupled convection. For a more detailed discussion see [5]. In both cases the solutions f and f^* and corresponding critical Rayleigh numbers R_c are well known. In the antisymmetric case f and f^* are identical

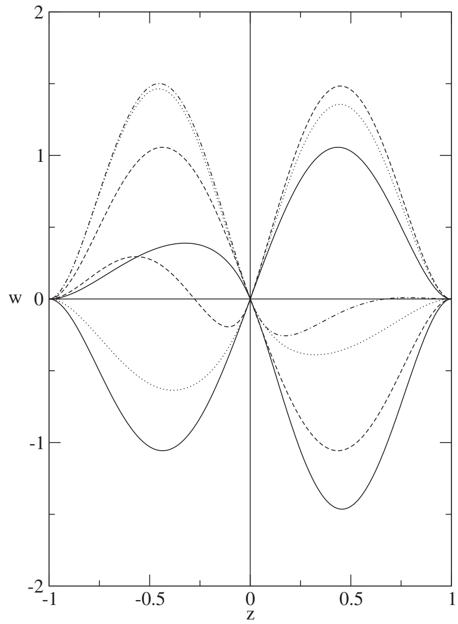

FIG. 1. Vertical velocities as functions of z for $R_c = 1100.6$, $a_c = 2.682$. The functions $w = f(z), f^*(z)$ that have a positive derivative at $z = 0$ have been obtained for $\kappa_0 = 1$ and arbitrary values of λ_0 and μ_0 (solid line), $\kappa_0 = \lambda_0 = 1/\sqrt{2}$, $\mu_0 = 2$ (dotted line), and for $\kappa_0 = \lambda_0 = 0.5$, $\mu_0 = 4$ (dashed line). The functions $w = f(z)$, $f^*(z)$ that have a negative derivative at $z = 0$ have been obtained for $\kappa_0 = 0.5$, $\lambda_0 = 2$, $\mu_0 = 1$ (solid line), $\kappa_0 = 1$, $\lambda_0 = 2$, $\mu_0 = 0.5$ (dashed line), $\kappa_0 = 2$, $\lambda_0 = 0.5$, $\mu_0 = 1$ (dotted line), and for $\kappa_0 = 2$, $\lambda_0 = 1$, $\mu_0 = 0.5$ (dash-dotted line).

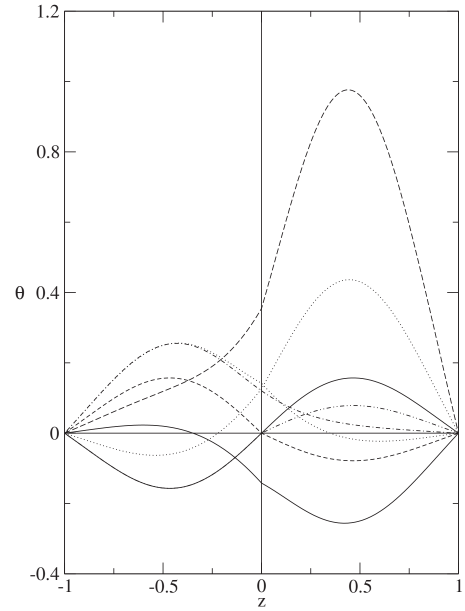

FIG. 2. Temperature profiles as functions of z for $R_c = 1100.6$, $a_c = 2.682$. The curves correspond to the same cases as shown in Fig. 1 except that the solid line with positive derivative at $z = 0$ requires $\kappa_0 = \lambda_0 = 1$ for $z > 0$ and becomes replaced by the dash-double-dotted curve for $\kappa_0 = 1$, $\lambda_0 = 2$. In both cases μ_0 remains arbitrary.

with the solution for the onset of convection in a single layer with a no-slip condition at one boundary and a stress-free condition at the other boundary, while the temperature is fixed on both boundaries. For this case $R_c = 1100.6$ corresponding to the wave number $a_c = 2.682$ [6].

In the symmetric or thermally coupled case the solutions f and f^* correspond to those describing the onset of convection in a single layer with two no-slip boundaries, one of which is thermally insulating, while the temperature is fixed on the other one. The critical value of the Rayleigh number in this case is $R_c = 1295.8$ corresponding to $a_c = 2.552$ [7]. Since this value of R_c does not exceed by much the value of the viscously coupled case we include it as physically relevant in our analysis.

B. Homologous solutions

It can be shown that the solution for $f(z), f^*(z)$ in the antisymmetric case of $\lambda_0 = \kappa_0 = \mu_0 = 1$ persists for arbitrary μ_0, λ_0 as long as $\kappa_0 = 1$ holds. This property is permitted by the matching conditions (12) and (13) because all even derivatives $f(z)$ and $f^*(z)$ vanish at $z = 0$. The antisymmetry with respect to $z = 0$ of the temperature perturbation is recovered if θ^* is multiplied by the factor λ_0. Examples of this type of solutions may be seen in Figs. 1 and 2. While $f(z)$ and $f^*(z)$ are independent of μ_0, λ_0, the corresponding temperature exhibits a discontinuity of its derivative at $z = 0$ for $\lambda_0 \neq 1$.

A more surprising result is the property that the same critical values and, in fact, the same $R(a)$ relationship is obtained for arbitrary values of μ_0, κ_0, and λ_0 as long as $\kappa_0 \lambda_0 \mu_0 = 1$ is fixed. The corresponding functions $f(z), f^*(z)$ vary strongly with μ_0 and λ_0 for $\kappa_0 = 1/(\lambda_0 \mu_0)$. Any symmetry with respect to $z = 0$ is now lost and θ and θ^* do no longer vanish at $z = 0$ as is evident from Fig. 2. As a result a tendency toward thermal coupling can be noticed as μ_0 increases.

The origin of the fact that the same $R(a)$ relationship holds for arbitrary values of λ_0 and μ_0 in the case $\kappa_0 = 1$ as well as in the case $\kappa_0 = 1/(\lambda_0 \mu_0)$ is as follows. The adjoint problem in the case $\kappa_0 = 1$ must have by definition a solution $\hat{f}(z), \hat{f}^*(z)$ corresponding to the same $R(a)$ relationship. It turns out that for arbitrary μ_0 and λ_0 this solution is provided by the solution of the original problem in the special case $1/\kappa_0 = \mu_0 \lambda_0$, i.e.,

$$\hat{f}(z) = \Delta^2 f(z), \quad \hat{f}^*(z) = \mu_0 \Delta^2 f^*(z). \tag{16}$$

The existence of a solution of the adjoint problem thus leads us to a new solution of the original problem. The reverse relationship also holds for arbitrary μ_0 and λ_0. The solution $\hat{f}(z), \hat{f}^*(z)$ of the adjoint problem in the case $1/\kappa_0 = \mu_0 \lambda_0$ is given by the same relationship (16) where the functions $f(z), f^*(z)$ represent the solutions of the original problem for $\kappa_0 = 1$.

An analogous situation is found in the case of the thermally coupled mode. The symmetric solution persists as long as $\kappa_0 \lambda_0 \mu_0 = 1$ is satisfied. In this case θ and θ^* and also $f(z)$ and $\mu_0 f^*(z)$ are symmetric with respect to $z = 0$. Conditions

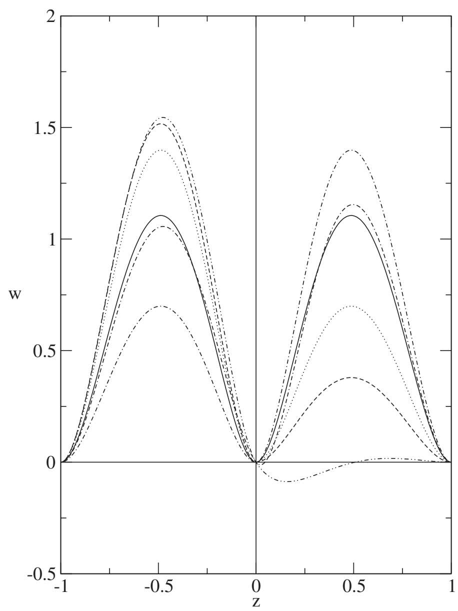

FIG. 3. Vertical velocities as functions of z for $R_c = 1295.6$, $a_c = 2.552$. The functions $w = f(z), f^*(z)$ have been obtained for $\kappa_0 = 1$, $\lambda_0 = 0.5$, $\mu_0 = 3$ (dash-double-dotted line) and $\kappa_0 = 1$, $\lambda_0 = 2.5$, $\mu_0 = 0.5$ (double-dash-dotted line). The solid curve is symmetric in z and corresponds to $\kappa_0 = 1$ with arbitrary values of λ_0 and μ_0. The dashed curve corresponds to $\kappa_0 = \lambda_0 = 0.5$, $\mu_0 = 4$. The dash-dotted curve corresponds to $\mu = 0.5$ and either $\kappa_0 = 1$, $\lambda_0 = 2$ or $\kappa_0 = 2$, $\lambda_0 = 1$. The dotted line corresponds to $\kappa_0 = \lambda_0 = 1/\sqrt{2}$, $\mu_0 = 2$.

(12) and (13) are satisfied in this way since all odd derivatives vanish at $z = 0$. Examples for such solutions are shown in Figs. 3 and 4.

Again it is found that the same neutral curve $R(a)$ also holds for arbitrary values of μ_0 and λ_0 as long as the condition $\kappa_0 = 1$ is satisfied because in this latter case the solution of the adjoint problem solves the original problem according to relationships (16). Examples for this case are displayed in Figs. 3 and 4. No simple symmetries are found in this case.

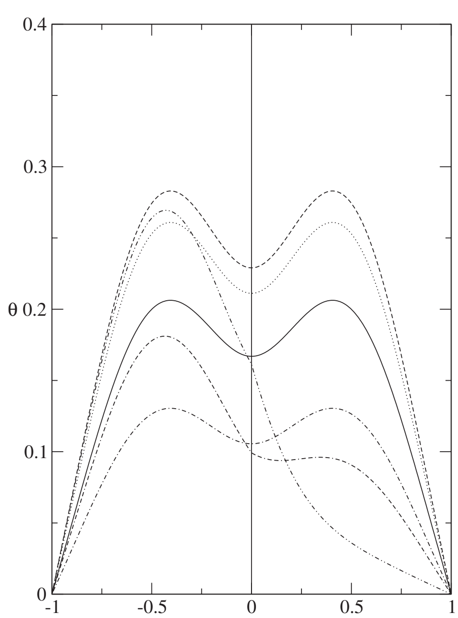

FIG. 4. Temperature profiles as functions of z for $R_c = 1295.6$, $a_c = 2.552$. The curves correspond to the same cases as shown in Fig. 3.

IV. ANALYSIS FOR THE CASE OF STRESS-FREE OUTER BOUNDARIES

When boundary conditions (10) are used, Eq. (9) for f and f^* can be solved explicitly in the form

$$f = A_1 \sin r(z+1) + A_2 \sin p_2(z+1) + \bar{A}_2 \sin \bar{p}_2(z+1), \tag{17a}$$

$$f^* = B_1 \sin r(z-1) + B_2 \sin p_2(z-1) + \bar{B}_2 \sin \bar{p}_2(z-1), \tag{17b}$$

where the bar indicates the complex conjugate. $r, p_2, \bar{p}_2$ correspond to the real and the two complex roots

$$r^2 = -a^2 + (a^2 R)^{1/3}, \tag{18a}$$

$$p_2^2 = -a^2 + (a^2 R)^{1/3}(-1 + i\sqrt{3})/2 \tag{18b}$$

of the cubic equation

$$(p^2 + a^2)^3 = a^2 R. \tag{19}$$

The second and third conditions of Eq. (12) can be used to eliminate $\bar{A}_2$ and $\bar{B}_2$,

$$\bar{A}_2 = (-A_1 \sin r - A_2 \sin p_2)/\sin \bar{p}_2,$$

$$\bar{B}_2 = (-B_1 \sin r - B_2 \sin p_2)/\sin \bar{p}_2. \tag{20}$$

The remaining four conditions of Eqs. (12) and (13) give rise to the following four equations:

$$A_1 c_1 + A_2 d_1 - B_1 c_1 - B_2 d_1 = 0, \tag{21a}$$

$$A_1 c_2 + A_2 d_2 + (B_1 c_2 + B_2 d_2)/(\kappa_0 \lambda_0) = 0, \tag{21b}$$

$$A_1 c_3 + A_2 d_3 + (B_1 c_3 + B_2 d_3)\mu_0 = 0, \tag{21c}$$

$$A_1 c_4 + A_2 d_4 - (B_1 c_4 + B_2 d_4)/\kappa_0 = 0, \tag{21d}$$

where the coefficients $c_n, d_n, n = 1, \ldots, 4$ are given by

$$c_1 = r \cos r - \bar{p}_2 \cot \bar{p}_2 \sin r, \tag{22a}$$

$$c_2 = (r^4 + 2a^2 r^2 - \bar{p}_2^4 - 2a^2 \bar{p}_2^2)\sin r, \tag{22b}$$

$$c_3 = (\bar{p}_2^2 - r^2)\sin r, \tag{22c}$$

$$c_4 = r(r^2 + a^2)^2 \cos r - \bar{p}_2(\bar{p}_2^2 + a^2)^2 \cot \bar{p}_2 \sin r, \tag{22d}$$

$$d_1 = p_2 \cos p_2 - \bar{p}_2 \cot \bar{p}_2 \sin p_2, \tag{22e}$$

$$d_2 = (p_2^4 + 2a^2 p_2^2 - \bar{p}_2^4 - 2a^2 \bar{p}_2^2)\sin p_2, \tag{22f}$$

$$d_3 = (\bar{p}_2^2 - p_2^2)\sin p_2, \tag{22g}$$

$$d_4 = p_2(p_2^2 + a^2)^2 \cos p_2 - \bar{p}_2(\bar{p}_2^2 + a^2)^2 \cot \bar{p}_2 \sin p_2. \tag{22h}$$

In order that a nontrivial solution of Eq. (21) exists, the determinant of the coefficient matrix must vanish. In the cases

$$\kappa_0 = 1 \tag{23a}$$

$$\text{and} \quad \kappa_0 = (\mu_0 \lambda_0)^{-1} \tag{23b}$$

the determinant assumes the relatively simple form

$$(c_1 d_4 - c_4 d_1)(c_3 d_2 - c_2 d_3)\left(\frac{1}{\kappa_0 \lambda_0} + 1\right)(\mu_0 + 1/\kappa_0). \tag{24}$$

Determinant (24) vanishes when either

$$(c_3 d_2 - c_2 d_3) = 0, \tag{25a}$$

$$\text{or} \quad (c_1 d_4 - c_4 d_1) = 0. \tag{25b}$$

In both cases the vanishing of the determinant does not depend on the ratios $\mu_0, \kappa_0, \lambda_0$ of material properties. Since one of these three parameters is fixed by either one of conditions (23) we thus conclude that the function $R(a)$ and, in particular, its critical value R_c for onset of convection do not depend on a two-dimensional subspace of the parameter space of the problem. Constraint (8) must be observed, of course, but this can be done through an appropriate choice of γ_0. Although ν_0 is another free parameter of the problem, we shall not include it in our count since it reflects the effect of the Prandtl number which represents the parameter that usually does not enter problems of onset of steady convection.

Condition (25a) for the vanishing of the determinant of the coefficient matrix of Eq. (21) can be satisfied through a particularly simple choice of R. Using the relationship

$$R(a) = (\pi^2 + a^2)^3 / a^2 \tag{26}$$

for ordinary Rayleigh-Bénard convection in the presence of stress-free boundaries one finds $r = \pi$ with the consequence $c_2 = c_3 = 0$. The important result is that Eq. (26) holds for all values of property ratios (3) as long as condition (8) and either Eq. (23a) or Eq. (23b) are satisfied. The well known critical value R_c minimizing expression (26) is given by

$$R_c = 27\pi^4/4 = 657.5 \quad \text{corresponding to} \quad a_c = \frac{1}{\sqrt{2}}. \tag{27}$$

In case (23a) we obtain as solution of Eq. (21)

$$A_1 = B_1, \quad A_2 = B_2 = 0, \tag{28}$$

which holds for arbitrary values of μ_0 and λ_0. This solution describes convection of the "viscous coupling" type. The vertical velocity is antisymmetric with respect to $z=0$ as are the temperature perturbations θ and θ^* when the latter is multiplied by the factor λ_0.

For the adjoint problem a simple solution comparable to Eq. (28) is found in the case $1/\kappa_0 = \mu_0 \lambda_0$. Using representation (17) for the functions $\hat{f}(z), \hat{f}^*(z)$ we find a manifold of solutions given by

$$\hat{A}_1 = \hat{B}_1/\mu_0, \quad \hat{A}_2 = \hat{B}_2 = 0. \tag{29}$$

There thus must exist a manifold of solutions of the original problem for arbitrary μ_0 and λ_0 with $1/\kappa_0 = \mu_0 \lambda_0$. This manifold is given by

$$A_1 = A_2 \frac{d_1 c_4 \lambda_0 (1 + \mu_0) - d_4 c_1 (1 + \lambda_0)}{c_1 c_4 (1 - \lambda_0 \mu_0)},$$

$$B_1 = A_2 \frac{d_1 c_4 (1 + 1/\mu_0) - d_4 c_1 (1 + \lambda_0)}{c_1 c_4 (1 - \lambda_0 \mu_0)},$$

$$B_2 = -A_2/\mu_0, \tag{30}$$

where the coefficients c_1, c_4, d_1, d_4 are determined by expressions (19), (22a), (22d), (22e), and (22h) together with expression (26). This solution also describes convection of the viscous coupling type. For $\kappa_0 = (\mu_0 \lambda_0)^{-1} \rightarrow 1$, when the expressions for A_1 and B_1 in Eq. (30) diverge, the solution becomes equal to solution (28). The dependence of w and θ on the parameters $\kappa_0, \lambda_0, \mu_0$ resembles that of the solutions plotted in Figs. 1 and 2. Of course, f'' and $f^{*\prime\prime}$ now vanish at the outer boundary instead of f' and $f^{*\prime}$. The solution $\hat{f}(z), \hat{f}^*(z)$ given by solution (30) according to expressions (16) solves the adjoint problem in the case $\kappa_0 = 1$.

In case (25b) we again obtain two classes of solutions. The simpler class is obtained this time for condition (23b),

$$A_2 = -A_1 c_1/d_1, \quad B_1 = -A_1/\mu_0, \quad B_2 = -A_2/\mu_0, \tag{31}$$

which describes the onset of thermally coupled convection in the two layers. In the particular case $\mu_0 = 1$ solution (31) describes the onset of convection in single fluid layer with stress-free condition and a fixed temperature on one boundary and a thermally insulating no-slip boundary on the other side. The critical conditions for this problem are known from the work [7],

$$R_c = 816.748 \quad \text{corresponding to} \quad a_c = 2.21. \tag{32}$$

There is thus no need to solve Eq. (25b) for R and a, although it is useful sometimes to have the entire neutral curve $R(a)$. For arbitrary μ_0 and λ_0 solution (31) corresponds to temperature perturbations that are symmetric with respect to the interface $z=0$. The same property is also exhibited by $f(z)$ and $\mu_0 f^*(z)$.

Again another manifold of more complex solutions corresponding to the same neutral curve $R(a)$ with the critical condition (32) is obtained when condition (23a) instead of Eq. (23b) is applied,

$$A_2 = A_1[d_1 c_2 c_3(\mu_0 - \lambda_0^{-1}) + c_1 d_2 c_3 (1 + \mu_0)/\lambda_0 - c_1 c_2 d_3 (\mu_0 + \mu_0/\lambda_0)]/N, \tag{33a}$$

$$B_1 = A_1[c_1 d_2 d_3 (\mu_0 - \lambda_0^{-1}) - d_1 c_2 d_3 (1 + \mu_0) + d_1 d_2 c_3 (1 + \lambda_0^{-1})]/N, \tag{33b}$$

$$B_2 = A_1[d_1 c_2 c_3(\mu_0 - \lambda_0^{-1}) - c_1 d_2 c_3(1+\mu_0) + c_1 c_2 d_3(1+\lambda_0^{-1})]/N, \quad (33c)$$

with

$$N \equiv c_1 d_2 d_3(\mu_0 - \lambda_0^{-1}) + d_1 c_2 d_3(1+\mu_0)/\lambda_0 - d_1 d_2 c_3 \mu_0(1+\lambda_0^{-1}). \quad (33d)$$

According to expressions (16) this manifold of solutions is related to the solutions of problem adjoint to the problem solved by solution (31). No obvious symmetries are exhibited by solutions (33). Similar to solution (28) that can be obtained as a special case of solution (30) in the limit $\lambda_0\mu_0 \to 1$, solution (31) is identical with the special case $\lambda_0\mu_0 = 1$ of solution (33).

V. CONCLUDING REMARKS

The examples discussed in this paper demonstrate how large manifolds of solutions may belong to a single eigenvalue relationship $R(a)$. In the particular problems investigated here it was not even necessary to derive those eigenvalue relationships since they were available from the literature on the onset of convection in a single fluid layer heated from below. Two two-dimensional manifolds of solutions have been identified for each of the four relationships $R(a)$ investigated in this paper. One of these two manifolds can easily be derived from symmetry considerations, but the other manifold could not be anticipated. Because of a relationship of form (16) between the solutions of the original problem and its adjoint one it has been possible to identify the second manifold of rather asymmetric solutions.

Although the particular mathematical properties of the system investigated in this paper depend on the identity of the two operators (9) in the two layers, it must be expected that the results found here will persist in an approximate sense when slight differences in the equations for the two layers are admitted. It may thus be possible to derive conditions for the onset of convection in superimposed fluid layers based on perturbations of solutions presented in this paper when Rayleigh numbers and heights of the layers do not differ significantly, while high contrasts between the material properties of the two fluids still exist.

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