Journal of Physics A: Mathematical and Theoretical

PAPER

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To cite this article: Niurka R Quintero *et al* 2019 *J. Phys. A: Math. Theor.* **52** 155401

IOP Publishing Journal of Physics A: Mathematical and Theoretical

J. Phys. A: Math. Theor. **52** (2019) 155401 (17pp) https://doi.org/10.1088/1751-8121/ab0dd9

Externally driven nonlinear Dirac equation revisited: theory and simulations

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Received 2 November 2018, revised 25 February 2019
Accepted for publication 7 March 2019
Published 21 March 2019

Abstract

The externally driven nonlinear Dirac (NLD) equation with scalar-scalar self-interaction studied in (2016 *J. Phys. A: Math. Theor.* **49** 065402) is revisited. By using a variational method and an ansatz with five collective coordinates, the dynamics of the NLD solitons is well described. It is shown that this new ansatz possesses certain advantages, namely the canonical momentum agrees with the field momentum, the energy associated to the collective coordinate equations agrees with the energy of the NLD soliton, whereas the ansatz with either three or four collective coordinates does not. Thus the study of the whole phase space of the system is enhanced. It is also shown that this approach is equivalent to the method of moments: the time variation of the charge, the momentum, the energy, and the first moment of the charge. The advantages of the new ansatz are illustrated by means of numerical simulations.

Keywords: nonlinear spinor field, collective coordinates, variational approach, soliton stability, external force, method of moments, numerical simulations

(Some figures may appear in colour only in the online journal)

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1. Introduction

The nonlinear Dirac (NLD) equation [1] appears in many fields of physics, ranging from nonlinear optics to quantum chromodynamics [2–8]. The NLD equation has solitary wave solutions (in short: solitons), which are spatially localized and have finite energy and charge [9]. In particular, for the NLD equation in $(1 + 1)$ dimensions (i.e. one time dimension plus one space dimension), several exact solitary wave solutions were derived for the quadratic nonlinearity [10, 11], for fractional nonlinearity [12], as well as for general nonlinearity [13–15]. The analytical expressions of the NLD solitons not only allow the interaction dynamics among them to be studied numerically [5, 16–19], but also provide a functional form of approximate solutions (ansatz) in order to study how the external potentials and forces affect the dynamics of the NLD solitons [20–23].

Specifically, in [22, 23] the externally driven NLD equation

$$(\mathrm{i}\gamma^{\mu}\partial_{\mu} - m)\Psi + g^2(\bar{\Psi}\Psi)\Psi = f(x)\,, \tag{1.1}$$

was considered, where $\Psi(x,t) = \{\Psi_1(x,t), \Psi_2(x,t)\}^T$ is a spinor with two components [14], and both the mass m and the coupling constant g are fixed parameters. Moreover, the following representation for the $1 + 1$ dimensional Dirac Gamma matrices: $\gamma^0 = \sigma_3$; $\gamma^1 = \mathrm{i}\sigma_2$, has been used, where σ_2 and σ_3 are the Pauli matrices. The function

$$f(x) = \begin{pmatrix} r_1 \mathrm{e}^{-\mathrm{i}K_1 x} \\ r_2 \mathrm{e}^{-\mathrm{i}K_2 x} \end{pmatrix} \tag{1.2}$$

is a two-component spinor inhomogeneous forcing term with real parameters r_j and K_j $(j = 1, 2)$.

For $f(x) = (0,\, 0)^T$, equation (1.1) is Lorentz invariant [19]. This fact is used to obtain the components of the soliton solution moving with a constant velocity v:

$$\begin{aligned} \Psi_1(x,t) &= [\cosh(\eta/2)A(x') + \mathrm{i}\sinh(\eta/2)B(x')]\,\mathrm{e}^{-\mathrm{i}\omega t'}, \\ \Psi_2(x,t) &= [\sinh(\eta/2)A(x') + \mathrm{i}\cosh(\eta/2)B(x')]\,\mathrm{e}^{-\mathrm{i}\omega t'}, \end{aligned} \tag{1.3}$$

where $\gamma = \cosh\eta = 1/\sqrt{1 - v^2}$ represents the Lorentz factor, $x' = \gamma(x - vt)$, and $t' = \gamma(t - vx)$. The localized functions $A(x)$ and $B(x)$ are defined by equation (2.4) in [22].

When $f(x) \neq (0, 0)^T$, an ansatz is assumed in [22] with three collective coordinates (CC) for the trial variational wave function. Owing to the smallness of the perturbation, the main modification to the exact solutions (1.3) to the NLD equation (equation (1.1) without driving forces) was that the parameters describing the position $q(t)$, the frequency $\omega(t)$, and phase $\phi(t)$ become time dependent unknown functions. In particular, the position of the solitary wave vt for the unforced case was replaced by $q(t)$. Moreover, the phase of the exact solution $\omega t' = \gamma\omega(t - vx)$ was replaced by $\phi(t) - \gamma\omega\dot{q}[x - q(t)]$ and $\omega(t)$ was considered a function of time.

By using this ansatz, from the variational method and Lagrange equation, the canonical momentum, $P_q(t)$, and the energy, $E_{\mathrm{cc}}(t)$, of the collective coordinates were obtained as a function of the collective coordinates. On the other hand by inserting this ansatz in the density of the field momentum and in the density of energy and by integrating over space, the field momentum, $P(t)$, and the energy, $E(t)$, of the NLD soliton were also obtained as a function of the three collective variables. However, expressions for the momenta were different and expressions for the energies were also different. The conditions $P(t) = P_q(t)$ and $E(t) = E_{\mathrm{cc}}(t)$ were guaranteed only at the initial time $t = 0$ and for a particular initial phase, $\phi(0)$ [22].

Consequently, the study of the dynamics of the soliton in [22] was restricted to this particular value of the initial phase. Recently, we noticed that another important consequence of the study reported in [22] is that the time variation of the energy $E(t)$ and the momentum $P(t)$ yields a different set of equations that should be satisfied by the collective variables. The main aim of the current study is to resolve this discrepancy. We demonstrate that this discrepancy disappears by using a new ansatz. We also show that this modification of the ansatz provides a considerable improvement of the collective coordinate theory:

(i) First, the expressions for the field momentum and for the canonical momentum as functions of the collective coordinates are the same, that is, $P(t) = P_q(t)$, for all $t \geqslant 0$.
(ii) Second, both energies also now have the same expression, that is, $E(t) = E_{\mathrm{cc}}(t)$, for all $t \geqslant 0$.
(iii) Third, the study of the whole phase space of the dynamical system is enhanced. One can set any value of the initial phase $\phi(0)$.
(iv) Finally, we show the equivalence between the variational method and the time variation of the charge, the momentum, the energy, and the first moment of the charge (method of moments).

For the perturbed sine-Gordon and φ^4 equations, nonlinear Schrödinger (NLS) equation, Thiele equation and other nonlinear equations supporting soliton solutions, the above discrepancy never appeared [24–28]. Indeed, it was shown from the variational method, from the method of moments and from the so-called Generalized Traveling Wave Method for the NLS equation and nonlinear Klein–Gordon equations, that the same equations of motion are obtained [24–26].

The rest of the paper is organized as follows: in section 2, an autonomous driven transformed NLD equation is briefly presented and the ansatz with five collective coordinates is introduced. By using the Lagrangian approach, the equations of motions are obtained. In the subsequent section, it is shown that the time variation of the charge, the momentum, the energy and the first moment of the charge result in the same equations of motion. In section 4, discussions on simulations of the driven NLD equation and on the numerical solutions of the collective coordinates equations are presented. Finally, concluding remarks are given in section 5. Some details of the calculations are shown in the appendices A and B.

2. Ansatz with five collective coordinates

We focus on the case in which the spatial period of the external force in (1.2) is determined by $K_1 = K_2 = K$. Using the transformation

$$\Psi(x,t) = \chi(x,t)\mathrm{e}^{-\mathrm{i}Kx}, \tag{2.1}$$

in (1.1), we obtain the autonomous driven transformed NLD equation

$$\mathrm{i}\gamma^\mu\partial_\mu\chi - m\chi + g^2(\bar{\chi}\chi)\chi = r - K\gamma^1\chi, \tag{2.2}$$

where $r = (r_1, r_2)^T$ is a two-component spinor. This equation can be obtained from the Euler–Lagrange equation

$$\partial_\mu\frac{\partial\tilde{\mathcal{L}}}{\partial(\partial_\mu\bar{\chi})} - \frac{\partial\tilde{\mathcal{L}}}{\partial\bar{\chi}} = 0, \tag{2.3}$$

where the Lagrangian density is determined by the expression

$$\tilde{\mathcal{L}} = \left(\frac{\mathrm{i}}{2}\right)[\bar{\chi}\gamma^\mu\partial_\mu\chi - \partial_\mu\bar{\chi}\gamma^\mu\chi] - m\bar{\chi}\chi + \frac{g^2}{2}(\bar{\chi}\chi)^2 - \bar{\chi}r - \bar{r}\chi + K\bar{\chi}\gamma^1\chi. \tag{2.4}$$

The corresponding adjoint transformed NLD equation reads

$$\mathrm{i}\partial_\mu\bar{\chi}\gamma^\mu + m\bar{\chi} - g^2(\bar{\chi}\chi)\bar{\chi} = -\bar{r} + K\bar{\chi}\gamma^1, \tag{2.5}$$

and can be derived in a similar way from the Euler–Lagrange equation:

$$\partial_\mu\frac{\partial\tilde{\mathcal{L}}}{\partial(\partial_\mu\chi)} - \frac{\partial\tilde{\mathcal{L}}}{\partial\chi} = 0. \tag{2.6}$$

In order to resolve the aforementioned discrepancy, a modified trial wave function is employed with five collective coordinates. In component form, this is given by:

$$\begin{aligned}\chi_1(x,t) &= \left(\cosh\left[\frac{\eta(t)}{2}\right]A(z) + \mathrm{i}\sinh\left[\frac{\eta(t)}{2}\right]B(z)\right)\mathrm{e}^{-\mathrm{i}\tilde{\phi}(t)+\mathrm{i}\frac{\tilde{p}(t)}{\cosh\eta(t)}z},\\ \chi_2(x,t) &= \left(\sinh\left[\frac{\eta(t)}{2}\right]A(z) + \mathrm{i}\cosh\left[\frac{\eta(t)}{2}\right]B(z)\right)\mathrm{e}^{-\mathrm{i}\tilde{\phi}(t)+\mathrm{i}\frac{\tilde{p}(t)}{\cosh\eta(t)}z},\end{aligned} \tag{2.7}$$

where $z = \cosh\eta\,[x - q(t)]$ and

$$A(z) = \frac{\sqrt{2}\beta(t)}{g}\frac{\sqrt{m+\omega(t)}\cosh[\beta(t)z]}{m+\omega(t)\cosh[2\beta(t)z]}, \tag{2.8}$$

$$B(z) = \frac{\sqrt{2}\beta(t)}{g}\frac{\sqrt{m-\omega(t)}\sinh[\beta(t)z]}{m+\omega(t)\cosh[2\beta(t)z]}. \tag{2.9}$$

Note that $\beta(t)$ is also time dependent due to $\beta(t) = \sqrt{m^2-\omega^2(t)}$. It is worth mentioning the main differences between this ansatz and the one introduced in [22]: here $\tilde{p}(t)$ and $\eta(t)$ are both *independent* variables, while in [22] $\tilde{p}(t) = \omega(t)\sinh\eta(t)$ and $\cosh\eta(t) = 1/\sqrt{1-\dot{q}^2(t)}$. For the sake of comparison with the previous results reported in [22], we use the notation $\tilde{p}(t)$ and $\tilde{\phi}(t)$ for the collective coordinates in the transformed NLD equation.

From equations (2.1) and (2.7), the ansatz corresponding to the original NLD equation (1.1) is given by

$$\Psi_1(z,t) = \left(\cosh\left[\frac{\eta(t)}{2}\right]A(z) + \mathrm{i}\sinh\left[\frac{\eta(t)}{2}\right]B(z)\right)\mathrm{e}^{-\mathrm{i}\phi(t)+\mathrm{i}\frac{p(t)}{\cosh\eta(t)}z}, \tag{2.10}$$

$$\Psi_2(z,t) = \left(\sinh\left[\frac{\eta(t)}{2}\right]A(z) + \mathrm{i}\cosh\left[\frac{\eta(t)}{2}\right]B(z)\right)\mathrm{e}^{-\mathrm{i}\phi(t)+\mathrm{i}\frac{p(t)}{\cosh\eta(t)}z}, \tag{2.11}$$

where

$$\tilde{p}(t) = p(t) + K, \tag{2.12}$$

$$\tilde{\phi}(t) = \phi(t) - Kq(t). \tag{2.13}$$

Curiously, using an ansatz with four collective coordinates, namely the position $q(t)$, the frequency $\omega(t)$, the phase $\tilde{\phi}(t)$ and $\tilde{p}(t)$, the expressions of the canonical momentum and the field momentum once again differ. Moreover, the energy corresponding to the dynamical system disagrees with the energy calculated inserting the above-mentioned four CC ansatz into equation (3.1). The details of these calculations are shown in the appendix B. Such a four CC ansatz assumes that $\cosh\eta(t) = \gamma = 1/\sqrt{1-\dot{q}^2(t)}$, therefore $\eta(t)$ depends on the velocity. This dependence is a consequence of the Lorentz invariance of the unperturbed NLD equation [19]. However, our NLD equation (1.1) is no longer Lorentz invariant. This may be the reason why we need to consider here $\eta(t)$ and $\dot{q}(t)$ as independent variables.

Inserting (2.7) into the Lagrangian density (2.4) and integrating over x we obtain

$$\tilde{L} = Q[\omega(t)]\left\{\tilde{p}(t)\dot{q}(t) + \dot{\tilde{\phi}}(t) - \tilde{p}(t)\tanh[\eta(t)] - \frac{\omega(t)}{\cosh[\eta(t)]}\right\} \\ - I_0[\omega(t)]\left\{\cosh\eta(t) - \dot{q}(t)\sinh\eta(t)\right\} + KQ[\omega(t)]\tanh[\eta(t)] - \tilde{U}[\omega(t),\eta(t),\tilde{\phi}(t),\tilde{p}(t)], \tag{2.14}$$

where

$$Q[\omega(t)] = \frac{2\beta(t)}{g^2\omega(t)},$$
$$I_0[\omega(t)] = M_0[\omega(t)] - \omega(t)Q[\omega(t)],$$
$$M_0[\omega(t)] = \frac{4m}{g^2}\operatorname{arctanh}(\alpha), \quad \alpha = \sqrt{[m-\omega(t)]/[m+\omega(t)]},$$

are calculated in the appendix A, together with the effective potential

$$\tilde{U}[\omega(t),\eta(t),\tilde{\phi}(t),\tilde{p}(t)] = \frac{2\pi\cos[\tilde{\phi}(t)](r_1\tilde{C} - r_2\tilde{S})}{g\sqrt{\omega(t)}\cosh[\eta(t)]\cosh[\tilde{a}\pi]}, \tag{2.15}$$

with $\tilde{a} = \tilde{p}(t)/\{2\beta(t)\cosh[\eta(t)]\}$. The functions $\tilde{C}$ and $\tilde{S}$ are given by

$$\tilde{C} = \cosh\left[\frac{\eta(t)}{2}\right]\cos(\tilde{b}) - \sinh\left[\frac{\eta(t)}{2}\right]\sin(\tilde{b}),$$
$$\tilde{S} = \sinh\left[\frac{\eta(t)}{2}\right]\cos(\tilde{b}) - \cosh\left[\frac{\eta(t)}{2}\right]\sin(\tilde{b}),$$

where $\tilde{b} = \tilde{a}\operatorname{arccosh}[m/\omega(t)]$.

In order to obtain the five equations of motion, we use the Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\tilde{L}}{\partial\dot{Y}_i} - \frac{\partial\tilde{L}}{\partial Y_i} = 0, \tag{2.16}$$

with $i = 1, 2, 3, 4, 5$. Specifically, for $Y_1 = q$, it is clear that the canonical momentum

$$\tilde{P}_q(t) = \frac{\partial\tilde{L}}{\partial\dot{q}} = \tilde{p}(t)Q[\omega(t)] + I_0[\omega(t)]\sinh[\eta(t)], \qquad \frac{\mathrm{d}\tilde{P}_q}{\mathrm{d}t} = 0 \tag{2.17}$$

is conserved. Moreover, by setting $Y_2 = \tilde{\phi}$, $Y_3 = \tilde{p}$, $Y_4 = \omega$, and $Y_5 = \eta$ in equation (2.16), we obtain the collective coordinate equations

$$\frac{\mathrm{d}Q}{\mathrm{d}t} + \frac{\partial\tilde{U}}{\partial\tilde{\phi}} = 0, \tag{2.18}$$

$$Q[\omega(t)]\{\dot{q}(t) - \tanh[\eta(t)]\} - \frac{\partial\tilde{U}}{\partial\tilde{p}} = 0, \tag{2.19}$$

$$\mathcal{M}=\left\{\tilde{p}(t)\dot{q}(t)+\dot{\tilde{\phi}}(t)-(\tilde{p}(t)-K)\tanh[\eta(t)]-\frac{\omega(t)}{\cosh[\eta(t)]}\right\}\frac{\mathrm{d}Q}{\mathrm{d}\omega}-Q[\omega(t)]$$
$$\sinh[\eta(t)]\{\dot{q}(t)-\tanh[\eta(t)]\}-\frac{\partial\tilde{U}}{\partial\omega}=0, \tag{2.20}$$

$$\mathcal{N}=-\frac{Q[\omega(t)]}{\cosh^2[\eta(t)]}\{\tilde{p}(t)-\omega(t)\sinh[\eta(t)]-K\}-I_0[\omega(t)]\left\{\sinh[\eta(t)]-\dot{q}(t)\cosh[\eta(t)]\right\}-\frac{\partial\tilde{U}}{\partial\eta}=0, \tag{2.21}$$

respectively. Using equations (2.17)–(2.21), straightforward calculations show that the energy of the dynamical system

$$\tilde{E}_{\mathrm{cc}}(t)=M_0[\omega(t)]\cosh[\eta(t)]+\{\tilde{p}(t)-K-\omega(t)\sinh[\eta(t)]\}Q[\omega(t)]\tanh[\eta(t)]+\tilde{U}(\omega,\eta,\tilde{\phi},\tilde{p}), \tag{2.22}$$

is a constant of motion. From the transformations (2.12), (2.13) and equations (2.17) and (2.22), it follows

$$\tilde{P}_q(t)=P_q(t)+KQ[\omega(t)], \tag{2.23}$$

$$\tilde{E}_{\mathrm{cc}}(t)=E_{\mathrm{cc}}(t). \tag{2.24}$$

3. Method of moments

Equation (2.2) is invariant under both time translation and space translation. Therefore, the total energy

$$\tilde{E}(t)=\int_{-\infty}^{+\infty}\mathrm{d}x\left\{-\frac{\mathrm{i}}{2}\left[\bar{\chi}\gamma^1\partial_x\chi-\partial_x\bar{\chi}\gamma^1\chi\right]+m\bar{\chi}\chi-\frac{g^2}{2}(\bar{\chi}\chi)^2+\bar{\chi}r+\bar{r}\chi-K\bar{\chi}\gamma^1\chi\right\}, \tag{3.1}$$

and the total momentum

$$\tilde{P}(t)=\int_{-\infty}^{+\infty}\mathrm{d}x\,\frac{\mathrm{i}}{2}\left[\bar{\chi}_x\gamma^0\chi-\bar{\chi}\gamma^0\chi_x\right], \tag{3.2}$$

are both conserved quantities, i.e.

$$\frac{\mathrm{d}\tilde{E}}{\mathrm{d}t}=0, \tag{3.3}$$

$$\frac{\mathrm{d}\tilde{P}}{\mathrm{d}t}=0, \tag{3.4}$$

whereas the charge

$$\tilde{Q}=\int_{-\infty}^{+\infty}\mathrm{d}x\,\bar{\chi}\,\gamma^0\chi \tag{3.5}$$

is not necessarily conserved. The details of the derivation of the continuity equations and of the conservation laws can be found in [22]. By substituting the ansatz (2.1) into equations (3.1), (3.2) and (3.5), first, we observe that the charge $Q(t)$ and the energy $E(t)$ of the NLD equation (1.1) are equal, respectively, to the charge $\tilde{Q}(t)$ and the energy $\tilde{E}(t)$ of the transformed equation. Second, the momentum of the NLD equation (1.1) satisfies $P(t)=\tilde{P}(t)-KQ(t)$.

Furthermore, the insertion of the ansatz (2.7) into the expressions for the field momentum (3.2) and the energy (3.1) leads, respectively, to

$$\tilde{P}(t) = \tilde{p}(t)Q[\omega(t)] + I_0[\omega(t)]\sinh[\eta(t)], \tag{3.6}$$

$$\tilde{E}(t) = M_0[\omega(t)]\cosh[\eta(t)] + \{\tilde{p}(t) - K - \omega(t)\sinh[\eta(t)]\}Q[\omega(t)]\tanh[\eta(t)] + \tilde{U}(\omega,\eta,\tilde{\phi},\tilde{p}). \tag{3.7}$$

From (2.17) and (3.6) it is clear that the momentum $\tilde{P}(t) = \tilde{P}_q(t)$, $\forall\, t \geqslant 0$. Hence, equation (3.4) is the same as equation (2.17). This implies, $P(t) = P_q(t)$, $\forall\, t \geqslant 0$, where $P(t)$ is the field momentum of the original system (1.1) and $P_q(t)$ its corresponding canonical momentum. This result solves the contradiction that appeared in [22], where the canonical momentum and the field momentum differed as functions of the collective coordinates. The condition $\mathrm{d}\tilde{P}/\mathrm{d}t = \mathrm{d}\tilde{P}_q/\mathrm{d}t = 0$ now provides a unique equation of motion. The comparison of equations (2.22) and (3.7), causes a similar observation for the energy of the transformed NLD equation $\tilde{E}(t)$ and the energy of the dynamical system $\tilde{E}_{\mathrm{cc}}(t)$, i.e. $\tilde{E}(t) = \tilde{E}_{\mathrm{cc}}(t)$ for all $t \geqslant 0$. As a consequence, $E(t) = E_{\mathrm{cc}}(t)$, $\forall\, t \geqslant 0$, where $E(t)$ is the energy of the original system (1.1) and $E_{\mathrm{cc}}(t)$ the energy corresponding to the ansatz (2.10), when the five variables $\{q(t), \phi(t), p(t), \omega(t), \eta(t)\}$ are used.

Multiplying equation (2.2) to the left by $\bar{\chi}$ and equation (2.5) to the right by χ, and adding both expressions, we obtain

$$\frac{\partial}{\partial t}(\bar{\chi}\gamma^0\chi) + \frac{\partial}{\partial x}(\bar{\chi}\gamma^1\chi) = \mathrm{i}\,[\bar{r}\chi - \bar{\chi}r]. \tag{3.8}$$

Integrating over x, and assuming that $\bar{\chi}(+\infty,t)\gamma^1\chi(+\infty,t) - \bar{\chi}(-\infty,t)\gamma^1\chi(-\infty,t) = 0$, we deduce that the evolution of the charge is governed by

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \mathrm{i}\int_{-\infty}^{+\infty}\mathrm{d}x\,[\bar{r}\chi - \bar{\chi}r]. \tag{3.9}$$

Inserting the ansatz (2.7) into the rhs of equation (3.9) and then, integrating, we show that

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{2\pi\sin[\tilde{\phi}(t)](r_1\tilde{C} - r_2\tilde{S})}{g\sqrt{\omega(t)}\cosh[\eta(t)]\cosh[\tilde{a}\pi]} = -\frac{\partial}{\partial\tilde{\phi}}\tilde{U}(\omega,\eta,\tilde{\phi},\tilde{p}), \tag{3.10}$$

which agrees with equation (2.18).

We now consider the evolution of the first moment of the charge. Multiplying equation (3.8) by x and integrating, we obtain

$$\frac{\mathrm{d}Q_1}{\mathrm{d}t} = \int_{-\infty}^{+\infty}\mathrm{d}x\,(\bar{\chi}\gamma^1\chi) + \mathrm{i}\int_{-\infty}^{+\infty}\mathrm{d}x\,x\,[\bar{r}\chi - \bar{\chi}r], \tag{3.11}$$

$$Q_1 = \int_{-\infty}^{+\infty}\mathrm{d}x\,x\,(\bar{\chi}\gamma^0\chi). \tag{3.12}$$

Inserting the ansatz (2.7) in equation (3.12) and integrating, the first moment of the charge simply reads

$$Q_1 = q(t)Q[\omega(t)]. \tag{3.13}$$

Its time evolution is determined by

$$\dot{q}(t)Q[\omega(t)] + q(t)\frac{\mathrm{d}Q}{\mathrm{d}t} = Q[\omega(t)]\tanh\eta + \mathrm{i}q(t)\int_{-\infty}^{\infty}\mathrm{d}x(\bar{r}\chi - \bar{\chi}r) + \frac{\partial\tilde{U}}{\partial\tilde{p}}, \tag{3.14}$$

which we combine with equation (3.9), this finally yields equation (2.19).

Deriving equation (3.7) with respect to time and taking into account equations (2.17)–(2.19), after some straightforward calculations we obtain:

$$\mathcal{M}\omega_t + \mathcal{N}\eta_t = 0. \tag{3.15}$$

One solution of this equation is $\mathcal{M} = 0$ and $\mathcal{N} = 0$, which agrees with equations (2.20) and (2.21). Therefore, the conservation of the energy provides the other two equations of motion.

4. Numerical simulations

The aim of this section is to show several other advantages of the current ansatz over the previous ansatz considered in [22]. Therefore, we compute the numerical solutions of the set of equations of motion for the collective coordinates and compare them with the results from the simulations of the externally driven NLD equation (1.1).

In order to solve the collective coordinate equations derived in section 2, we substitute $\dot{q}$ from equation (2.19) into equation (2.21), and obtain an algebraic equation for the collective coordinates. Therefore, by fixing initial values for η, q, ω and $\tilde{\phi}$ from this algebraic equation, the initial condition for $\tilde{p}$ and the value of the conserved momentum $\tilde{P}_q = \tilde{p}(0)Q[\omega(0)] + I_0[\omega(0)]\sinh[\eta(0)]$ are then obtained. Once the initial conditions have been fixed, the equations of motion (2.17)–(2.21) can be solved by a Mathematica program and then the evolution of the five collective coordinates is obtained.

The driven NLD equation given in equations (1.1) and (1.2) is a PDE for which various numerical schemes have been proposed (see [19] for a review of them). It is also reported there that the operator splitting method performs better than the other schemes in terms of accuracy and efficiency (the readers are referred to [19] and [29] for a detailed description of the method). Here we adopt the computational domain $[-100, 100]$, (i.e. $L = 100$), the time step $\Delta t = 0.025$, and final times $t_{fin} = 210$ or $t_{fin} = 800$. For the initial condition we use equations (2.10) and (2.11) by specifying the initial values $\eta(0)$, $q(0)$, $\omega(0)$, $p(0) = \tilde{p}(0) - K$ and $\phi(0) = \tilde{\phi}(0) + Kq(0)$, i.e.

$$\begin{aligned}
\Psi_1(x,0) = &\left(\cosh\left[\frac{\eta(0)}{2}\right]A\left(\cosh[\eta(0)][x-q(0)]\right) + \mathrm{i}\sinh\left[\frac{\eta(0)}{2}\right]\right.\\
&\left.\times B\left(\cosh[\eta(0)][x-q(0)]\right)\right)\mathrm{e}^{-\mathrm{i}\phi(0)+\mathrm{i}p(0)[x-q(0)]},\\
\Psi_2(x,0) = &\left(\sinh\left[\frac{\eta(0)}{2}\right]A\left(\cosh[\eta(0)][x-q(0)]\right) + \mathrm{i}\cosh\left[\frac{\eta(0)}{2}\right]\right.\\
&\left.\times B\left(\cosh[\eta(0)][x-q(0)]\right)\right)\mathrm{e}^{-\mathrm{i}\phi(0)+\mathrm{i}p(0)[x-q(0)]}.
\end{aligned}$$

To facilitate the comparison with the results reported in [22], the parameters are set to $g = 1$, $m = 1$, $r_1 = r = 0.01$, $r_2 = 0$, and $K = n\pi/L$, where n is an integer number $n \in [-6, 6]$. The values of $\lambda = 2\pi/K$ should be much larger than the width of the soliton, otherwise the forces acting on different parts of the soliton will differ.

As reported in [22], under the action of the driving force the NLD soliton moves, such that the soliton position $q(t)$ oscillates around a mean trajectory $\bar{v}t$, where $\bar{v}$ is the mean velocity.

Table 1. Frequency of intrinsic soliton oscillations and average soliton velocity as a function of the parameter $K = n\pi/100$. We compare the results of the 5-CC ansatz and numerical simulation at various values of K in multiples of $\pi/100$. Here we choose the initial conditions for the CC variables $q(0) = 0$, $\eta(0) = \text{arctanh}\,(0.1)$, $\omega(0) = 0.9$, $\tilde{\phi}(0) = \phi(0) = \pi/2$, $\tilde{p}(0) = \omega(0)\sinh[\eta(0)] + K\,(p(0) = \tilde{p}(0) - K)$. Other parameters are: $m = 1$, $g = 1$, $r_1 = r = 0.01$, and $r_2 = K_2 = 0$. The results of the 3-CC ansatz are referred to table 1 in [22] and included here to facilitate comparison.

n	Ω_{3cc}	Ω_{5cc}	Ω_{sim}	$\bar{v}_{3cc}$	$\bar{v}_{5cc}$	$\bar{v}_{sim}$
−6	0.911	0.916	0.911	0.0855	0.107	0.100
−5	0.911	0.916	0.911	0.0865	0.106	0.100
−4	0.911	0.908	0.911	0.0914	0.103	0.100
−3	0.903	0.908	0.903	0.102	0.0998	0.100
−2	0.903	0.900	0.903	0.111	0.0965	0.100
−1	0.895	0.900	0.895	0.115	0.0940	0.100
0	0.888	0.900	0.895	0.115	0.0926	0.0999
1	0.888	0.892	0.895	0.113	0.0925	0.0999
2	0.880	0.892	0.887	0.110	0.0934	0.0999
3	0.880	0.892	0.887	0.106	0.0950	0.0998
4	0.880	0.884	0.880	0.103	0.0968	0.0997
5	0.872	0.884	0.880	0.0999	0.0987	0.0997
6	0.872	0.877	0.880	0.0972	0.1003	0.0996

We compute the discrete Fourier transform (DFT) of $q(t) - \bar{v}t$ for the CC theory and the simulations and observe the same frequencies as above for the soliton charge. Table 1 shows the comparison of the results from the simulations reported in [22] and the results from the numerical solutions for the five collective coordinates. Using an ansatz with three collective coordinates, the agreement between the mean velocity computed from the 3-CC theory, $\bar{v}_{3cc}$, and the mean velocity computed from the simulations of the NLD equation, $\bar{v}_{sim}$, was not as good (the maximal relative error, $|\bar{v}_{sim} - \bar{v}_{3cc}|/\bar{v}_{sim}$, for the above parameters reached 15.12%). The reason for this discrepancy is not solely the fact that the plane wave phonons are excluded from the 3-CC theory. The new ansatz also neglects the phonons, nevertheless the maximal error between the mean velocity computed from the 5-CC theory, $\bar{v}_{5cc}$, and $\bar{v}_{sim}$ is now 7.41%. Moreover, the maximal relative error between Ω_{5cc} and Ω_{sim} is now 0.56%, while this error in [22] was 0.91%.

In figures 1–4, we show a comparison between simulations of the NLD equation and numerical solutions of the collective coordinates for sets of initial conditions, where either $\phi(0) = \tilde{\phi}(0) = 0$ or $\phi(0) = \tilde{\phi}(0) = \pi$. These figures represent four representative cases: (i) in figure 1, the soliton does not move, although the charge oscillates; (ii) figure 2 shows a moving soliton due to the initial condition; (iii) in figure 3, the soliton moves owing to the perturbation ($K \neq 0$); and (iv) in figure 4, the soliton moves faster due to the initial condition and to the perturbation. In all simulations, it has first been verified that the energy is conserved and equal to its initial value and second, that the quantity $P(t) + KQ(t)$ is also conserved and equal to $P(0) + KQ(0)$.

In the left-hand panel of figure 1, $Q(t)$ obtained from the simulations of equation (1.1) oscillates with two frequencies. The Discrete Fourier Transform (DFT) of $Q(t)$ shows two peaks at $\omega_1 = 0.928$ (highest peak) and at $\omega_2 = 0.987$ (second peak corresponding to the phonon frequency, see [22]). In the right-hand panel, $Q(t)$ from the numerical solutions of the collective coordinates oscillates with only one frequency $\omega_{cc} = 0.911(\approx \omega_1)$. This frequency

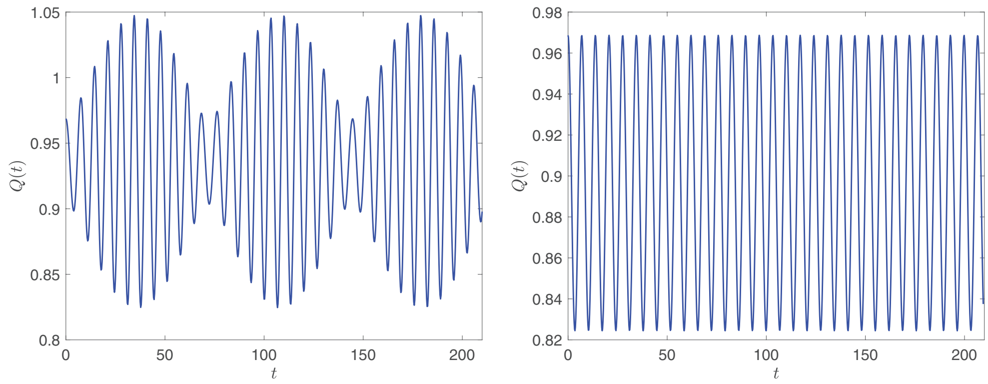

Figure 1. Evolution of the charge. Left panel: simulations of the driven NLD equation (1.1). Right-hand panel: numerical solutions of the CC equations. ICs: $\omega(0) = 0.9,\ q(0) = 0,\ \tilde{\phi}(0) = \phi(0) = \pi,\ \eta(0) = 0,\ \tilde{p}(0) = p(0) = 0$. Parameters of equations (1.1) with (1.2): $m = 1$, $g = 1$, $r_1 = r = 0.01$, $r_2 = 0$ and $K = 0$.

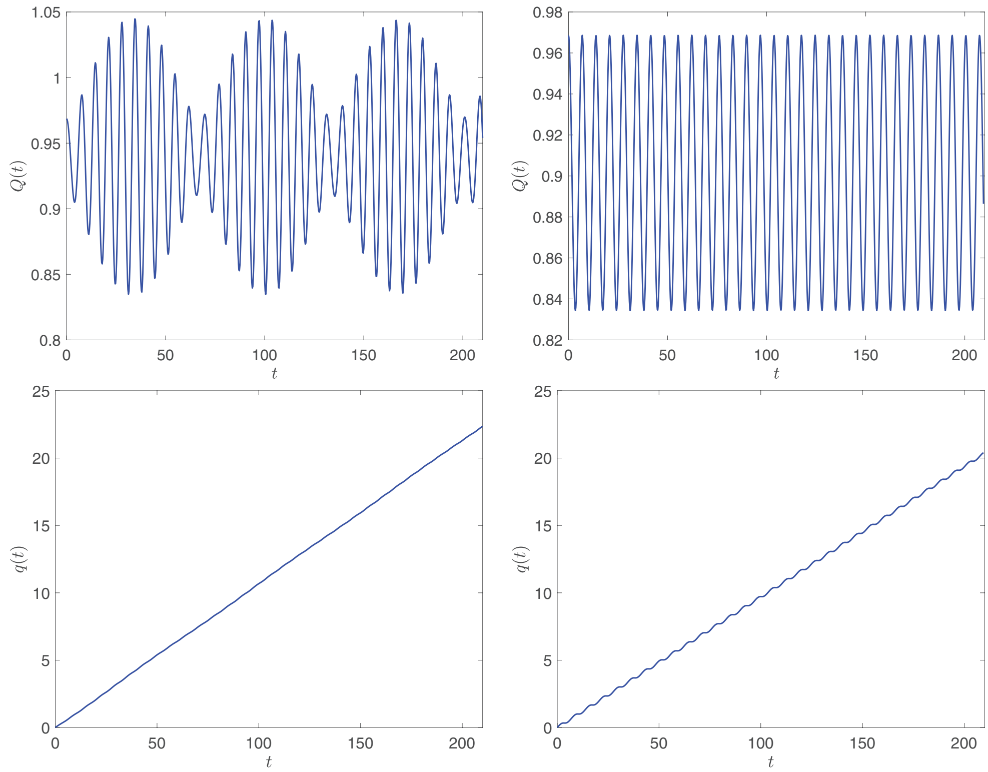

Figure 2. Evolution of the charge $Q(t)$ and of the position of the center of the soliton $q(t)$. The soliton travels to the right with a mean velocity. Left panels: results from simulations of the driven NLD equation (1.1). Right-hand panels: results from the 5-CC theory. ICs: $q(0) = 0,\ \omega(0) = 0.9,\ \tilde{\phi}(0) = \phi(0) = \pi,\ \eta(0) = \text{arctanh}(0.1)$, $\tilde{p}(0) = p(0) = 0.0896$. Parameters as in figure 1.

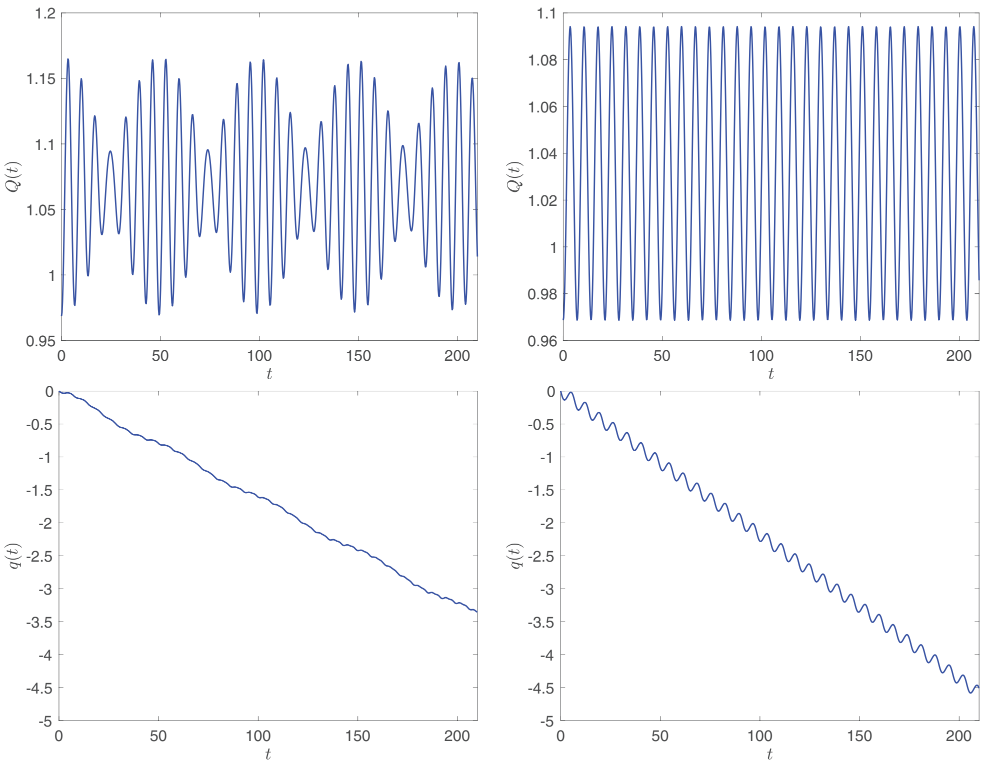

Figure 3. Evolution of the charge $Q(t)$ and the position of the center of the soliton $q(t)$. Left panel: results from simulations of the driven NLD equation (1.1). Right-hand panel: results from the 5-CC theory. ICs: $q(0) = 0$, $\omega(0) = 0.9$, $\tilde{\phi}(0) = \phi(0) = 0$, $\eta(0) = 0$, $\tilde{p}(0) = 0.184$ ($p(0) = \tilde{p}(0) - K$). Parameters as in figure 1, except $K = 6\pi/100$.

is close to the initial value of $\omega(0)$, while ω_2 corresponds to the phonon frequency. For the considered parameters, the soliton does not move, the energy is conserved and is equal to $E(t) = E_{\rm cc}(t) = 0.868$. Moreover, since $K = 0$, the momentum of the NLD equation (1.1) is also conserved and is equal to $P(t) = P_q(t) = 0$.

In figure 2, the case in which the soliton moves due to the initial conditions is considered. The value of K is also zero, and the other parameters are also fixed as in figure 1, except here $\eta(0) = \operatorname{arctanh}(0.1)$. The oscillations of the charge $Q(t)$ of the NLD equation and of the CC theory are similar to those observed in the previous figure. Indeed, $Q(t)$ from CC oscillates with only one frequency $\omega = 0.898$, while the DFT of $Q(t)$ from the simulations of the NLD equation shows the highest peak at $\omega_1 = 0.898 (= \omega)$ and of the second peak at $\omega_2 = 0.987$. For this set of initial conditions, $\bar{v}_{5\rm CC} = 0.0969$, which is very close to $\bar{v} = 0.107$ observed in the simulations. In the simulations, we have verified that the momentum and the energy of the transformed system is conserved and equal to $P(t) = 0.0931$ and $E(t) = 0.876$, respectively.

The soliton can also move due to a constant force. In figure 3, we start from the static soliton solution and we observe that the soliton moves to the left with a mean velocity $\bar{v} = -0.0162$, whereas $Q(t)$ oscillates with two frequencies, $\omega_1 = 0.898$ and $\omega_2 = 1.017$. This evolution of the center of the soliton is predicted by the CC theory, where $q(t)$ oscillates with the frequency ω_1 around a mean trajectory with $\bar{v}_{5\rm CC} = -0.0218$ (see $q(t)$ in the right lower panel). As in the previous cases, in the CC theory $Q(t)$ oscillates with only one frequency $0.898 (= \omega_1)$. In this

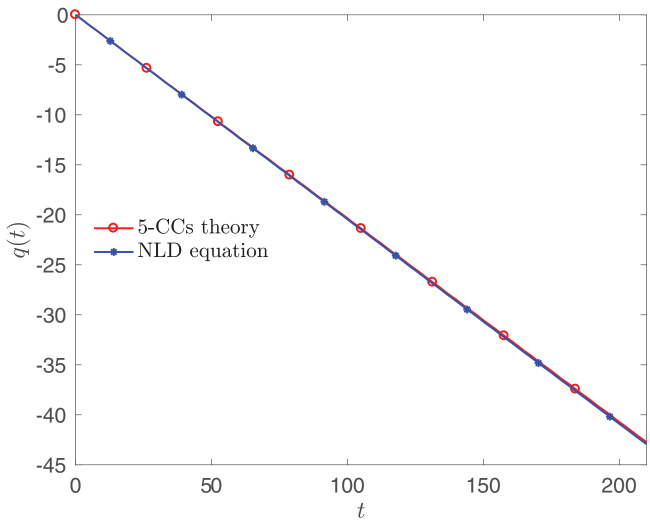

Figure 4. Evolution of $q(t)$: the results from the simulations of the NLD equations (1.1) with (1.2) are represented by a blue line with stars, and the results from the numerical solutions of the 5-CC theory are represented by a red line with circles. ICs: $\omega(0) = 0.9$, $q(0) = 0$, $\tilde{\phi}(0) = \phi(0) = 0$, $\eta(0) = -0.197$, $\tilde{p}(0) = 0$ ($p(0) = -K$). Other parameters as in figure 3.

case, it is verified that the energy is conserved $E(t) = E_{\rm cc}(t) = 0.988$. The momentum $P(t)$ is not conserved because $K \neq 0$, however the quantity $P(t) + KQ(t) = P_q(0) + KQ(0) = 0.1$ 78 is verified.

The action of the perturbation and the initial conditions can be used in order to obtain a faster soliton. Indeed, in figure 4, setting $K = 6\pi/100$ and $\eta(0) = -0.197$, we observe that in the CC theory $q(t)$ oscillates with a frequency close to 0.898 around a mean trajectory with $\bar{v}_{5\rm CC} = -0.204$. This prediction is confirmed by the numerical simulations of equations (1.1) with (1.2), where a mean velocity $\bar{v}_{\rm sim} = -0.205$ and a frequency 0.898 is obtained.

5. Conclusions

We have revisited the dynamics of solitary waves in the externally driven nonlinear Dirac equation. The driving is assumed periodic in space. After a given transformation, this system is simplified and becomes an autonomous driven transformed nonlinear Dirac equation, in which there is no longer explicit dependence on space. Therefore, in the transformed equation, not only is the energy $\tilde{E}(t)$ conserved, but also the momentum $\tilde{P}(t)$.

For the autonomous driven transformed nonlinear Dirac equation, we assumed an ansatz with five independent collective coordinates, namely the mass center $q(t)$, the momentum $\tilde{p}(t)$, the phase $\tilde{\phi}(t)$, and the frequency $\omega(t)$ of the soliton together with a collective coordinate $\eta(t)$. Using this ansatz and the variational approach, we derived the canonical momentum $\tilde{P}_q(t)$ and the energy $\tilde{E}_{\rm cc}(t)$ as functions of the collective coordinates. Both magnitudes are conserved and, moreover, they agree with the field momentum $\tilde{P}(t)$ and the energy of the soliton $\tilde{E}(t)$, respectively. This is the main result of the current research. In this way, we resolve the discrepancy reported in [22, 23], where the two momenta and the two energies differ. We have also shown that by using four collective coordinates this discrepancy persists. Different momenta and energies yielded two different sets of equations of motion for the collective coordinates. As a consequence, the uniqueness of the dynamics of the solitary wave under the driving was broken.

Another advantage of the new ansatz is that we can consider the cases where the initial phase is not fixed to $\pi/2$. The phase $\tilde{\phi}(0) = \pi/2$ was fixed in [22] so that the canonical momentum would be equal to the field momentum at the initial time $t = 0$. Here, this restriction is not necessary. In the current research, numerical computations have been performed for $\tilde{\phi}(0) = \phi(0) = 0$ or π, where good agreement between numerical solutions of the collective coordinates and simulations of the driven NLD equation was obtained. Our numerical results show four representative cases in which: (i) the action of a constant force ($K = 0$) does not move the initial static soliton; (ii) the soliton moves due to its initial velocity under the action of a constant force ($K = 0$); (iii) the spatial periodic force $K \neq 0$ moves the initial static soliton; and (iv) the initial soliton moves faster due to the initial velocity and to the action of an inhomogeneous force.

Moreover, we show that by using this ansatz, the time variation of the charge, the field momentum, the energy, and the first moment of the charge yield the same five equations of motions as using the variational approach. This second method (method of moments) is less general than the variational approach since *a priori* it remains unknown which set of moments should be used.

Acknowledgments

We acknowledge financial support from the Ministerio de Economía y Competitividad of Spain through FIS2014-54497-P (NRQ) and MTM2015-65888-C4-1-P (RAN), Junta de Andalucía through FQM-262 (RAN), and Feder Funds (European Union) (NRQ and RAN). NRQ acknowledges the financial support from the Alexander von Humboldt Foundation and the hospitality of the Physikalisches Institut at the University of Bayreuth (Germany). SS acknowledges the financial support from the National Natural Science Foundation of China (Nos. 11822102, 11421101). FGM acknowledges financial support and hospitality of the University of Seville.

Appendix A. Relevant integrals

The Lagrangian (2.14) is obtained after inserting the ansatz (2.7) into the following expressions and integrating

$$\int_{-\infty}^{+\infty} \mathrm{d}x\, \mathrm{i}\, \frac{[\bar{\chi}\gamma^{\mu}\partial_{\mu}\chi - \partial_{\mu}\bar{\chi}\gamma^{\mu}\chi]}{2} = Q[\omega(t)] \left\{ \tilde{p}(t)\dot{q}(t) + \dot{\tilde{\phi}}(t) - \tilde{p}(t)\tanh[\eta(t)] \right\} - I_0[\omega(t)] \times \{\cosh[\eta(t)] - \dot{q}(t)\sinh[\eta(t)]\}, \tag{A.1}$$

$$\int_{-\infty}^{+\infty} \mathrm{d}x \left[-m\bar{\chi}\chi + \frac{g^2}{2}(\bar{\chi}\chi)^2 \right] = -\frac{M_0[\omega(t)]}{\cosh[\eta(t)]} + \frac{I_0[\omega(t)]}{\cosh[\eta(t)]}, \tag{A.2}$$

$$-\frac{\mathrm{i}}{2}\int_{-\infty}^{+\infty} \mathrm{d}x \left[\bar{\chi}\gamma^1\partial_x\chi - \partial_x\bar{\chi}\gamma^1\chi\right] = I_0[\omega(t)]\cosh[\eta(t)] + \tilde{p}(t)Q[\omega(t)]\tanh[\eta(t)] \tag{A.3}$$

$$\int_{-\infty}^{+\infty} \mathrm{d}x\,(\bar{\chi}r + \bar{r}\chi) = \frac{2\pi\cos[\tilde{\phi}(t)](r_1\tilde{C} - r_2\tilde{S})}{g\sqrt{\omega(t)}\cosh[\eta(t)]\cosh[\tilde{a}\pi]} = \tilde{U}[\omega(t),\eta(t),\tilde{\phi}(t),\tilde{p}(t)], \tag{A.4}$$

$$\int_{-\infty}^{+\infty} \mathrm{d}x\,(\bar{\chi}\gamma^1\chi) = \tanh[\eta(t)]\int_{-\infty}^{+\infty} \mathrm{d}z\,[A^2(z) + B^2(z)] = Q[\omega(t)]\tanh[\eta(t)], \tag{A.5}$$

where

$$Q[\omega(t)] = \int_{-\infty}^{+\infty} \mathrm{d}x(\bar{\chi}\gamma^0\chi) = \int_{-\infty}^{+\infty} \mathrm{d}z[A^2(z) + B^2(z)] = \frac{2\beta(t)}{g^2\omega(t)}, \tag{A.6}$$

$$I_0[\omega(t)] = \int_{-\infty}^{+\infty} \mathrm{d}z(B'A - A'B) = \frac{2}{g^2}\left[2m\,\mathrm{arctanh}(\alpha) - \beta(t)\right], \tag{A.7}$$

$$M_0[\omega(t)] = m\int_{-\infty}^{+\infty} \mathrm{d}x\,(\bar{\chi}\chi) = m\int_{-\infty}^{+\infty} \mathrm{d}z\,(A^2 - B^2) = \frac{4m}{g^2}\,\mathrm{arctanh}(\alpha). \tag{A.8}$$

From (A.6)–(A.8), it is clear that

$$I_0[\omega(t)] = M_0[\omega(t)] - \omega(t)Q[\omega(t)], \tag{A.9}$$

where $M_0[\omega(t)]$ is the mass in the rest frame. In order to calculate $\tilde{U}[\omega(t),\eta(t),\tilde{\phi}(t),\tilde{p}(t)]$ we use

$$\begin{aligned} J(\omega,\eta,\tilde{\phi},\tilde{p}) &= \int_{-\infty}^{+\infty} \mathrm{d}zA(z)\cos[2\beta(t)\tilde{a}z] = \frac{\sqrt{2[m+\omega(t)]}\beta(t)}{g\omega(t)}\int_{-\infty}^{+\infty} \mathrm{d}z\frac{\cosh[\beta(t)z]\cosh(\mathrm{i}2\beta(t)\tilde{a}z)}{\frac{m}{\omega(t)} + \cosh[2\beta(t)z]} \\ &= \frac{\sqrt{2[m+\omega(t)]}\beta(t)}{g\omega(t)}\int_0^{+\infty} \mathrm{d}z\frac{\cosh[(1+\mathrm{i}2\tilde{a})\beta(t)z] + \cosh[(1-\mathrm{i}2\tilde{a})\beta(t)z]}{\frac{m}{\omega(t)} + \cosh[2\beta(t)z]} = \frac{\pi\cos[\tilde{b}]}{g\sqrt{\omega(t)}\cosh[\tilde{a}\pi]}, \end{aligned} \tag{A.10}$$

where $\tilde{a} = \tilde{p}/\{2\beta(t)\cosh[\eta(t)]\}$ and $\tilde{b} = \tilde{a}\,\mathrm{arccosh}[m/\omega(t)]$. In a similar way, we obtain

$$N(\omega,\eta,\tilde{\phi},\tilde{p}) = \int_{-\infty}^{+\infty} \mathrm{d}zB(z)\sin[2\beta(t)\tilde{a}z] = \frac{\pi\sin[\tilde{b}]}{g\sqrt{\omega(t)}\cosh[\tilde{a}\pi]}. \tag{A.11}$$

Appendix B. Four collective coordinates ansatz

In this section we use a four collective coordinates ansatz as an approximatted solution of equation (2.2)

$$\begin{aligned} \chi_1(x,t) &= \left(\cosh\left[\frac{\eta(t)}{2}\right]A(z) + \mathrm{i}\sinh\left[\frac{\eta(t)}{2}\right]B(z)\right)\mathrm{e}^{-\mathrm{i}\tilde{\phi}(t) + \mathrm{i}\frac{\tilde{p}(t)}{\cosh\eta(t)}z}, \\ \chi_2(x,t) &= \left(\sinh\left[\frac{\eta(t)}{2}\right]A(z) + \mathrm{i}\cosh\left[\frac{\eta(t)}{2}\right]B(z)\right)\mathrm{e}^{-\mathrm{i}\tilde{\phi}(t) + \mathrm{i}\frac{\tilde{p}(t)}{\cosh\eta(t)}z}, \end{aligned} \tag{B.1}$$

where $z = \cosh\eta\,[x - q(t)]$, and $A(z)$ and $B(z)$ are given by equations (2.8) and (2.9), respectively. Here $\eta(t)$ is not an independent collective coordinate, and is given by

$$\cosh\eta = \frac{1}{\sqrt{1-\dot{q}^2}} \equiv \gamma[\dot{q}(t)]. \tag{B.2}$$

This implies that $\tanh\eta = \dot{q}$ and $\sinh\eta = \dot{q}\gamma$.

In order to obtain the equations of motion we insert (B.1) into the Lagrangian density (2.4), integrate over x and obtain

$$\bar{L} = Q[\omega(t)]\dot{\tilde{\phi}}(t) - \frac{M_0[\omega(t)]}{\gamma[\dot{q}(t)]} + KQ[\omega(t)]\dot{q}(t) - \bar{U}[\omega(t), \tilde{\phi}(t), \tilde{p}(t), \dot{q}(t)], \tag{B.3}$$

where the effective potential $\bar{U}[\omega(t), \tilde{\phi}(t), \tilde{p}(t), \dot{q}(t)]$ is just the potential $\tilde{U}[\omega(t), \eta(t), \tilde{\phi}(t), \tilde{p}(t)]$ given by equation (2.15) setting the condition (B.2).

The equations of motion are obtained by using the Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\bar{L}}{\partial\dot{Y}_i} - \frac{\partial\bar{L}}{\partial Y_i} = 0, \tag{B.4}$$

with $i = 1, 2, \cdots 4$. Starting with $Y_1 = q$, it is clear that the canonical momentum

$$\bar{P}_q(t) = M_0[\omega(t)]\dot{q}(t)\gamma[\dot{q}(t)] + KQ[\omega(t)] - \frac{\partial\bar{U}}{\partial\dot{q}}[\omega(t), \tilde{\phi}(t), \tilde{p}(t), \dot{q}(t)], \tag{B.5}$$

is conserved, i.e. $d\bar{P}_q/\mathrm{d}t = 0$. Moreover, by setting $Y_2 = \tilde{\phi}$, $Y_3 = \tilde{p}$ and $Y_4 = \omega$, in equation (B.4), we obtain the collective coordinate equations

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = -\frac{\partial\bar{U}}{\partial\tilde{\phi}}, \tag{B.6}$$

$$\frac{\partial\bar{U}}{\partial\tilde{p}} = 0, \tag{B.7}$$

$$\left\{\frac{\omega(t)}{\gamma[\dot{q}(t)]} - \dot{\tilde{\phi}}(t) - K\dot{q}(t)\right\}\frac{\mathrm{d}Q}{\mathrm{d}\omega} = -\frac{\partial\bar{U}}{\partial\omega}, \tag{B.8}$$

respectively. Multiplying equation (B.5) by $\ddot{q}$, equation (B.6) by $\dot{\tilde{\phi}}$, equation (B.7) by $\dot{\tilde{p}}$ and equation (B.8) by $\dot{\omega}$ and adding all the equations, we obtain that the energy is constant, i.e. $d\bar{E}_{\mathrm{cc}}/\mathrm{d}t = 0$, where

$$\bar{E}_{\mathrm{cc}}(t) = P_q(t)\dot{q}(t) + \frac{M_0[\omega(t)]}{\gamma[\dot{q}(t)]} - KQ[\omega(t)]\dot{q}(t) + \bar{U}[\omega(t), \tilde{\phi}(t), \tilde{p}(t), \dot{q}(t)]. \tag{B.9}$$

On another hand, in section 3 it is shown that regardless of the ansatz the total energy (3.1) and the total momentum (3.2) are both conserved quantities. Inserting the 4-CC ansatz (B.1) into equations (3.2) and (3.1) we obtain

$$\bar{P}(t) = \tilde{p}(t)Q[\omega(t)] + I_0[\omega(t)]\dot{q}(t)\gamma[\dot{q}(t)], \tag{B.10}$$

$$\bar{E}(t) = M_0[\omega(t)]\gamma[\dot{q}(t)] + \{\tilde{p}(t) - K - \omega(t)\dot{q}(t)\gamma[\dot{q}(t)]\}Q[\omega(t)]\dot{q}(t) + \bar{U}[\omega(t), \tilde{\phi}(t), \tilde{p}(t), \dot{q}(t)], \tag{B.11}$$

respectively. Substracting equations (B.10) and (B.5), we obtain that the canonical momentum and the field momentum are not equal. In particular,

$$\bar{P}(t) - \bar{P}_q(t) = \{\tilde{p}(t) - K - \omega(t)\dot{q}(t)\gamma[\dot{q}(t)]\}Q[\omega(t)] + \frac{\partial \bar{U}}{\partial \dot{q}}[\omega(t), \tilde{\phi}(t), \tilde{p}(t), \dot{q}(t)], \tag{B.12}$$

indicates that taking 4 CC ansatz $\bar{P}(t) \neq \bar{P}_q(t)$. Moreover, the difference of $\bar{E}(t)$ and $\bar{E}_{cc}(t)$ is

$$\bar{E}(t) - \bar{E}_{cc}(t) = \{\bar{P}(t) - \bar{P}_q(t)\}\dot{q}(t), \tag{B.13}$$

which in general is not zero. Therefore, from the variational approach and from the method of moments different sets of CC equations are obtained. The discrepancy that we have found when an ansatz with 3 CC is used persists when an ansatz with four collective coordinates is employed.

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